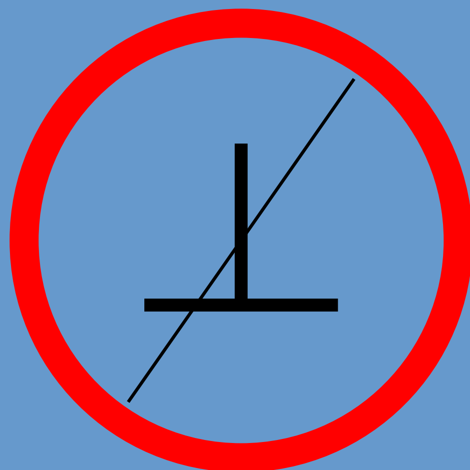
Stop thinking about bottoms when writing programs . . .



Thorsten Altenkirch
University of Nottingham

Trouble with $\bot$

Trouble with $\perp$

$$(*) :: \mathbb{N} \to \mathbb{N} \to \mathbb{N}$$

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$$x * y = y * x \qquad ?$$

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$$x * y = y * x \qquad ?$$

No, because

$$\begin{aligned} 0 * \perp &= 0 \\ \perp * 0 &= \perp \end{aligned}$$

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- Programs are **not** natural phenomena. . .

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- Do we actually care about non-terminating programs?
- Programs are **not** natural phenomena. . .
- Programs are **constructed**!

Do we need $\perp$ to be lazy?

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$$from :: \mathbb{N} \to [\mathbb{N}]$$

$$from\ n = n : (from\ (n+1))$$

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- *from* is total, **if** we interpret lists as a terminal coalgebra.

$$[A] = \nu X.1 + A \times X$$

data vs codata

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$$evenLength :: [\,a\,] \rightarrow \mathbf{Bool}$$

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- Problem:

$$evenLength\ (from\ 0) = \bot$$

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  $$\begin{aligned} \mathit{from} &:: \mathbb{N} \to [a]^{\omega} \\ \mathit{evenLength} &:: [\,a\,] \to \textbf{Bool} \end{aligned}$$
- $\mathit{evenLength}\ (\mathit{from}\ 0)$ doesn't typecheck.

Can we always avoid $\perp$?

Can we always avoid $\bot$?

$$\textbf{data SK} = \mathrm{S} \mid \mathrm{K} \mid \textbf{SK} :@\ \textbf{SK}$$

$$
\begin{aligned}
&nf :: \textbf{SK} \to \textbf{SK}\\
&nf\ \mathrm{S} &&= \mathrm{S}\\
&nf\ \mathrm{K} &&= \mathrm{K}\\
&nf\ (t :@\ u) &&= (nf\ \ t)@(nf\ \ u)
\end{aligned}
$$

$$
\begin{aligned}
&(@) :: \textbf{SK} \to \textbf{SK} \to \textbf{SK}\\
&\mathrm{K} && @t\ \ = \mathrm{K} :@\ t\\
&(\mathrm{K} :@\ t) && @u = t\\
&\mathrm{S} && @t\ \ = \mathrm{S} :@\ t\\
&(\mathrm{S} :@\ t) && @u = (\mathrm{S} :@\ t) :@\ u\\
&((\mathrm{S} :@\ t) :@\ u)@v = (t@v)@(u@v)
\end{aligned}
$$

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- We cannot implement

  $$pos :: \mathbb{R} \rightarrow \mathbf{Bool}$$

- Indeed, all total computable functions of type $\mathbb{R} \rightarrow \mathbf{Bool}$ are constant (Brouwer).
- However, there are perfectly reasonable partial implementations of pos.

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We need $\bot$ for:

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- more examples ?

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- I am going to show how we can fix this. . .
- without making Epigram partial.

Monads. . .

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- A monad $m :: * \to *$ is given by

  $$\begin{array}{lll} return & :: & a \to m\ a \\ (\geqslant) & :: & (m\ a) \to (a \to m\ b) \to m\ b \end{array}$$

  subject to some equations.

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  subject to some equations.

- We can use monads to *encapsulate* effects (e.g. state)

$$\begin{aligned} &newIORef\ :: a \to \mathbf{IO}\ (\mathbf{IORef}\ a) \\ &readIORef :: \mathbf{IORef}\ a \to \mathbf{IO}\ a \\ &writeIORef :: \mathbf{IORef}\ a \to a \to \mathbf{IO}\ () \end{aligned}$$

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- and to *model* effects (e.g. state) :

  $$\begin{aligned} &\mathbf{data\ ST}\ s\ a = \mathrm{M}\ (s \to (a, s)) \\ &\mathbf{instance}\ \mathit{Monad}\ (\mathbf{ST}\ s)\ \mathbf{where} \\ &\quad \mathit{return}\ a \qquad = \mathrm{M}\ (\lambda s \to (a, s)) \\ &\quad (\mathbf{ST}\ f) \gg\!= g = \mathrm{M}\ (\lambda s \to \mathbf{let}\ (a, s') = f\ s \\ &\qquad\qquad\qquad\qquad\qquad\qquad \mathrm{M}\ g' = g\ a \\ &\qquad\qquad\qquad\qquad\qquad \mathbf{in}\ g'\ s') \end{aligned}$$

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$$\bot :: \mathbf{D}\ a$$

$$\bot = \mathrm{Later}\ \bot$$

Iteration with Delay

$$rep :: (a \rightarrow \mathbf{D}\ (\mathit{Either}\ b\ a)) \rightarrow a \rightarrow \mathbf{D}\ b$$

$$rep\ k\ a = k\ a \gg\!= \lambda ba \rightarrow$$

$$\quad \mathbf{case}\ ba\ \mathbf{of}$$

$$\qquad \mathrm{Left}\ b \rightarrow \mathrm{Now}\ b$$

$$\qquad \mathrm{Right}\ a \rightarrow \mathrm{Later}\ (rep\ k\ a)$$

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&\mathit{race} :: (\mathbf{D}\ a) \to (\mathbf{D}\ a) \to (\mathbf{D}\ a)
\end{aligned}
$$

$$
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&\mathit{race}\ (\mathrm{Now}\ a)\ \ \_ &&= \mathrm{Now}\ a \\
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- We have $\mathit{rec}\ f \simeq f\ (\mathit{rec}\ f)$
if f is ω-continuous,
however all definable f are.

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- $\mathbf{P}\ a$ and hence $a \to \mathbf{P}\ b$ are ωCPOs.
- $rec\ f = \sqcup_{i \in \mathbb{N}} f^i \perp$ the code before constructs $\sqcup$ in $a \to \mathbf{P}\ b$.

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- Still to do: recursive datatypes by a constructive implementation of the standard domain-theoretic construction.

Thank you

- Thanks to Conor McBride & the Epigram Team
(James Chapman, Peter Morris, Wouter Swierstra)
see `www.e-pig.org` for more information on Epigram.
- Acknowledgements to Tarmo Uustalu and Venanzio Capretta
for joint work on a partial paper…
- Looking for my papers?
Type “Thorsten” into google…