

# Recolouring Graph Colourings

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joint work with

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after an idea of

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BCTCS 2006, Swansea

# Introduction

A **proper vertex  $k$ -colouring** of a graph  $G = (V, E)$  is

- a function  $\alpha : V \rightarrow \{1, 2, \dots, k\}$
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- **nodes** the proper vertex  $k$ -colourings of  $G$
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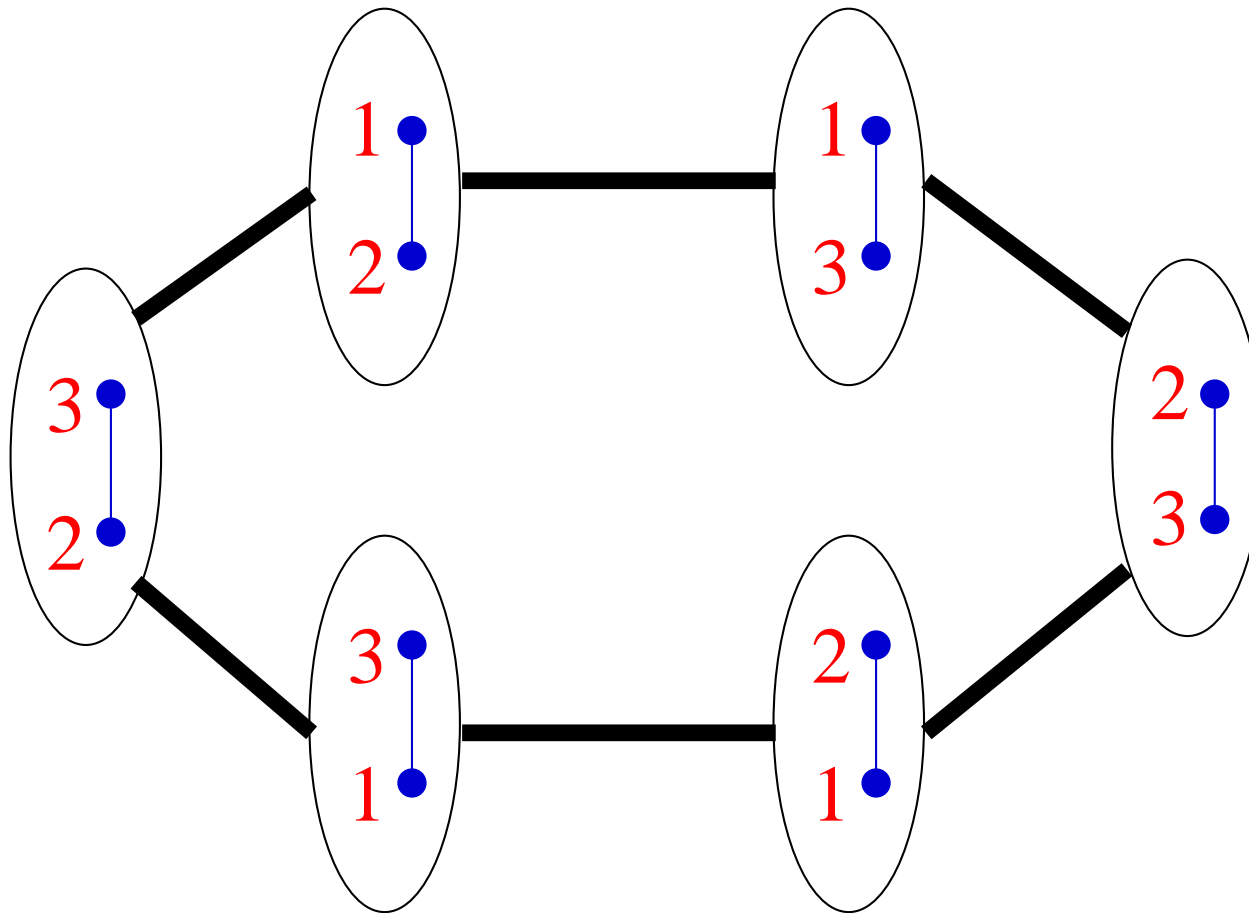
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- **nodes** the proper vertex  $k$ -colourings of  $G$
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If  $\mathcal{C}_k(G)$  is **connected** we say  $G$  is  **$k$ -mixing**

# An example

The 3-colour graph of  is



# Two decision problems

## $k$ -MIXING

**Instance:**  $k$ -colourable graph  $G$

**Question:** Is  $G$   $k$ -mixing?

## $k$ -COL-PATH

**Instance:** Graph  $G$ ,  $k$ -colourings  $\alpha$  and  $\beta$  of  $G$

**Question:** Is there a path between  $\alpha$  and  $\beta$  in  $\mathcal{C}_k(G)$ ?

# A sufficiency condition

The colouring number (or degeneracy) of a graph  $G$  is

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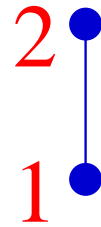
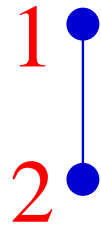
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This is **best possible**:

there exist graphs that are not  $(\text{col}(G) + 1)$ -mixing

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## Theorem

- 3-chromatic graphs are never 3-mixing
- For  $\chi \geq 4$ , there exist
  - $\chi$ -chromatic graphs that are  $\chi$ -mixing, and
  - $\chi$ -chromatic graphs that are not  $\chi$ -mixing

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$$w(\overrightarrow{uv}, \alpha) = \begin{cases} +1 & \text{if } uv \text{ coloured } 12, 23 \text{ or } 31 \\ -1 & \text{if } uv \text{ coloured } 21, 32 \text{ or } 13 \end{cases}$$

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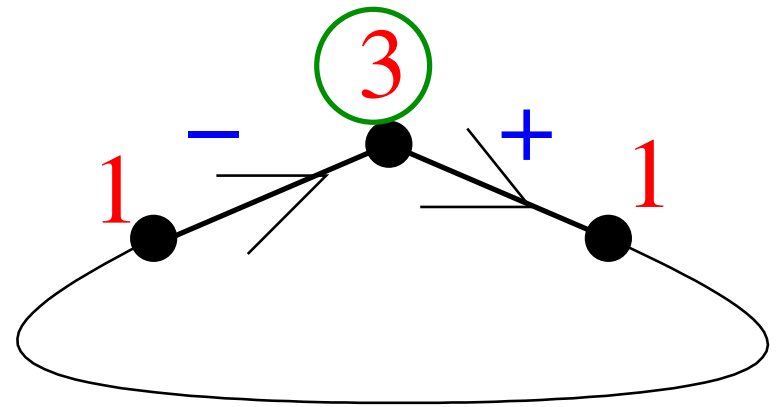
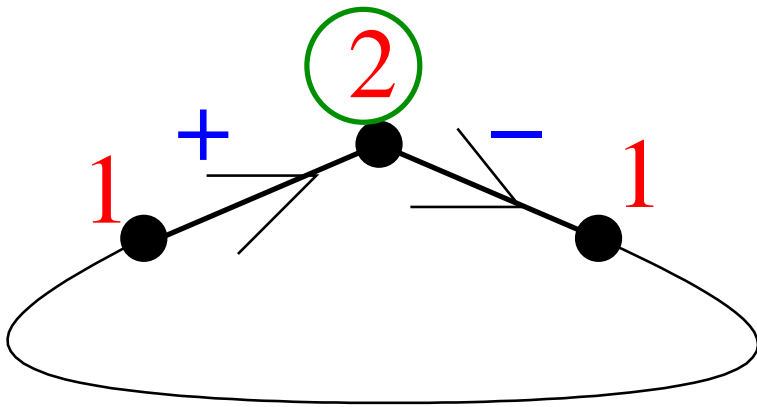
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- For a **directed cycle**  $\vec{C}$ , define

$$W(\vec{C}, \alpha) = \sum_{\vec{e} \in \vec{C}} w(\vec{e}, \alpha)$$

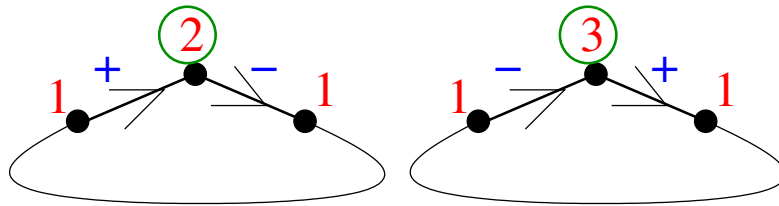
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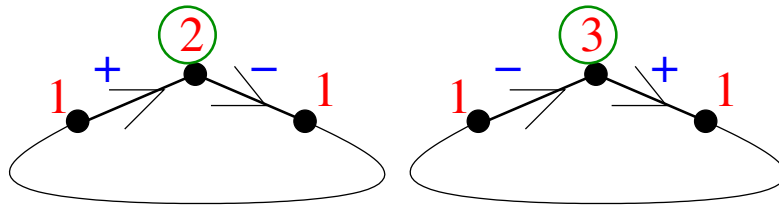
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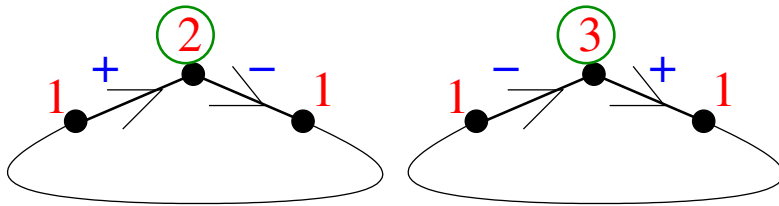


- From  $\alpha$ , define the 3-colouring  $\beta$  by

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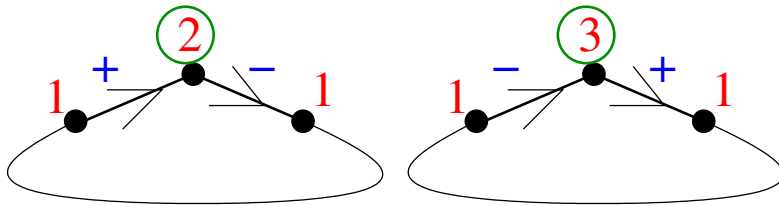
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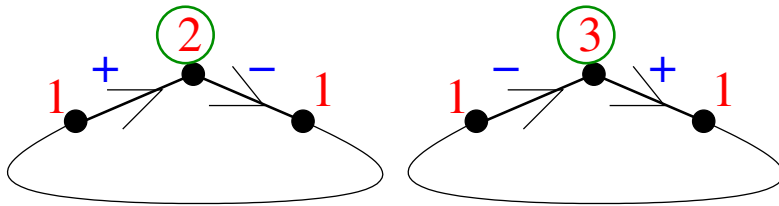
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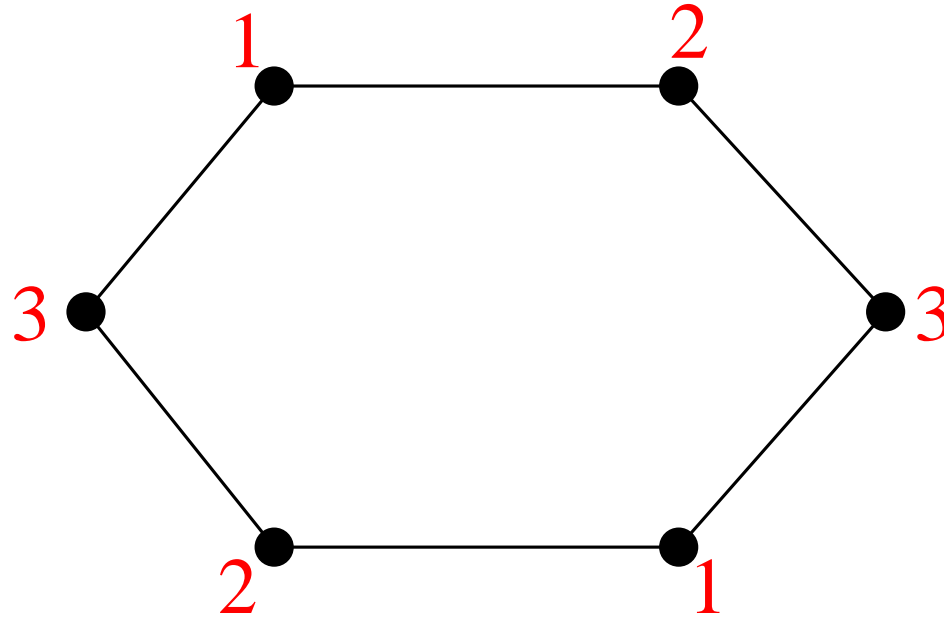
- All **edge weights change sign** under  $\beta$
- For  $C$  an odd cycle in  $G$ ,  $W(\vec{C}, \alpha) = -W(\vec{C}, \beta) \neq 0$
- Hence  $\alpha$  and  $\beta$  are not connected in  $\mathcal{C}_3(G)$

# 3-MIXING and 3-COL-PATH

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## Theorem

- Given two 3-colourings  $\alpha, \beta$  of a graph  $G$ , it can be **decided in polynomial-time** whether or not  $\alpha$  and  $\beta$  are **connected** by a path in  $\mathcal{C}_3(G)$   
That is, 3-COL-PATH  $\in$  P

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- Given  $k$ -colourings  $\alpha$  and  $\beta$  of  $G$ :  
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This is **best possible**:

for all  $\chi, k$ , with  $k \geq \chi$ , there exists a  $\chi$ -chromatic graph with  $k$ -colourings  $\alpha$  and  $\beta$ , for which  $\chi - 2$  extra colours are not enough to recolour from  $\alpha$  to  $\beta$

# Open Questions

- What can be said about the complexity of  $k$ -MIXING and  $k$ -COL-PATH, for  $k \geq 4$ ?
- Is 3-MIXING coNP-complete?
- Perhaps 3-MIXING is in P?