



Reversible Quantum Circuits from Irreversible functions

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Reversible Circuits

- Landauer's Principle states:
“any logically irreversible manipulation of information, such as the erasure of a bit or the merging of two computation paths, must be accompanied by a corresponding entropy increase in non-information bearing degrees of freedom of the information processing apparatus or its environment”



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“any logically irreversible manipulation of information, such as the erasure of a bit or the merging of two computation paths, must be accompanied by a corresponding entropy increase in non-information bearing degrees of freedom of the information processing apparatus or its environment”
- Reversing a computation puts the computer back into its initial state.

Classical Reversible Circuits



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- The Not gate.

$$x_0 \text{ --- } \boxed{\text{Not}} \text{ --- } \neg x_0$$

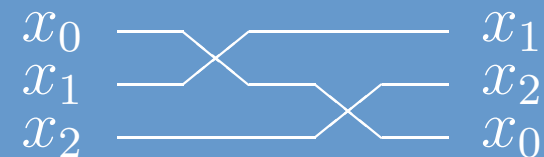
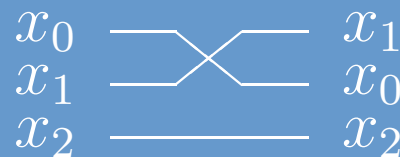


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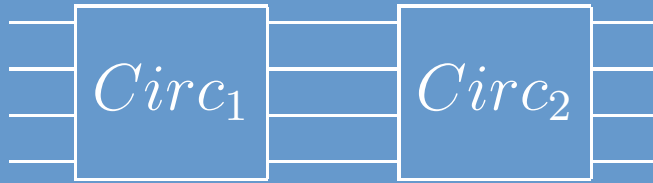
- Wire permutations.



Classical Reversible Circuits



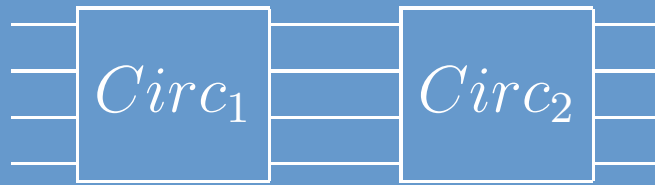
- Sequential composition.



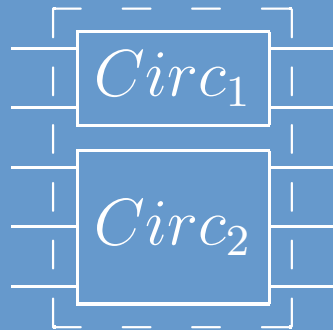
Classical Reversible Circuits



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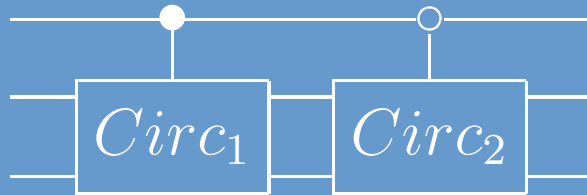
- Parallel composition.



Classical Reversible Circuits



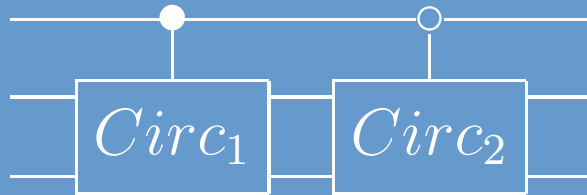
- The conditional operation.



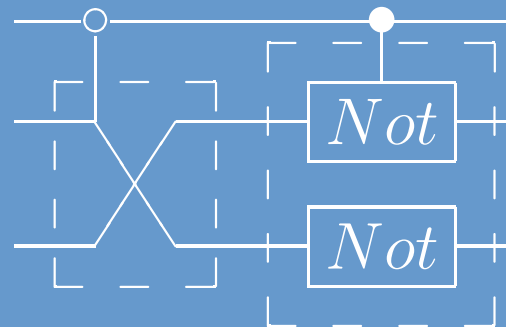
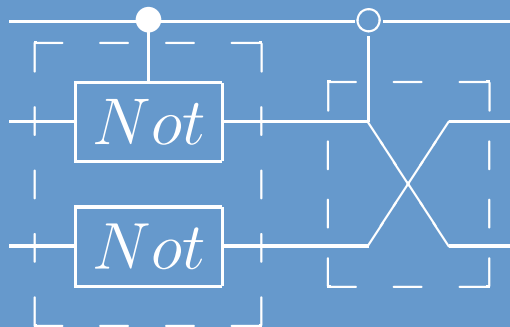
Classical Reversible Circuits



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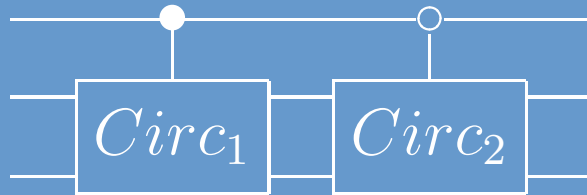
- Example:



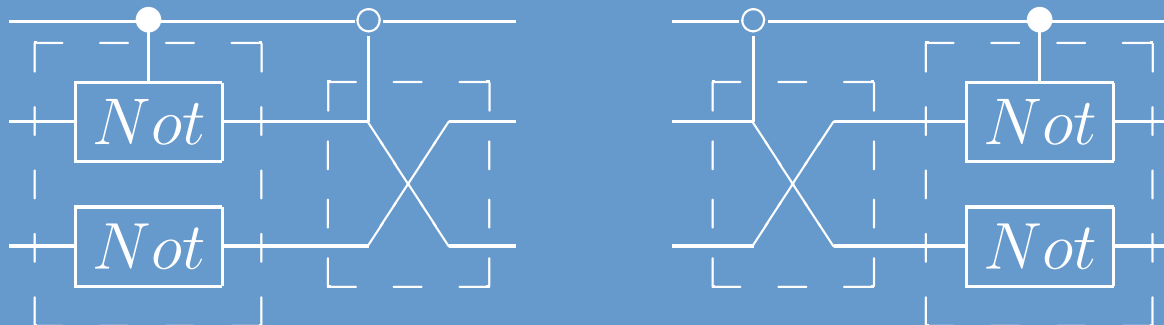
Classical Reversible Circuits



- The conditional operation.



- Example:



- But what about Quantum Circuits?



Classical to Quantum

- Classical reversible circuits act on bits.

$$\text{ReversibleCircuits} \in [\text{Bool}]_n \rightarrow [\text{Bool}]_n$$



Classical to Quantum

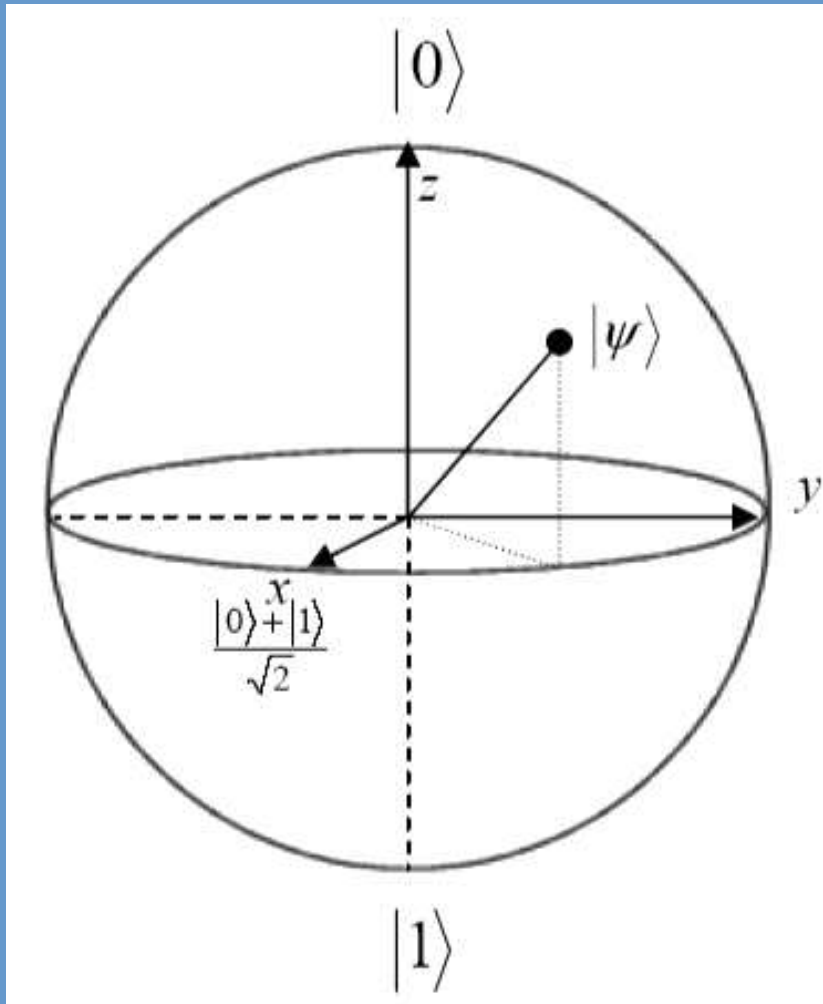
- Classical reversible circuits act on bits.
 $ReversibleCircuits \in [Bool]_n \rightarrow [Bool]_n$
- Quantum circuits act on qubits.

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$$

$$|\alpha|^2 + |\beta|^2 = 1$$



The Bloch Sphere





Quantum Circuits

- The classical reversible circuits are a subset of Quantum circuits.



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- The Rotation gate.

$$\text{---} \boxed{Rot\ u} \text{---}$$



Quantum Circuits

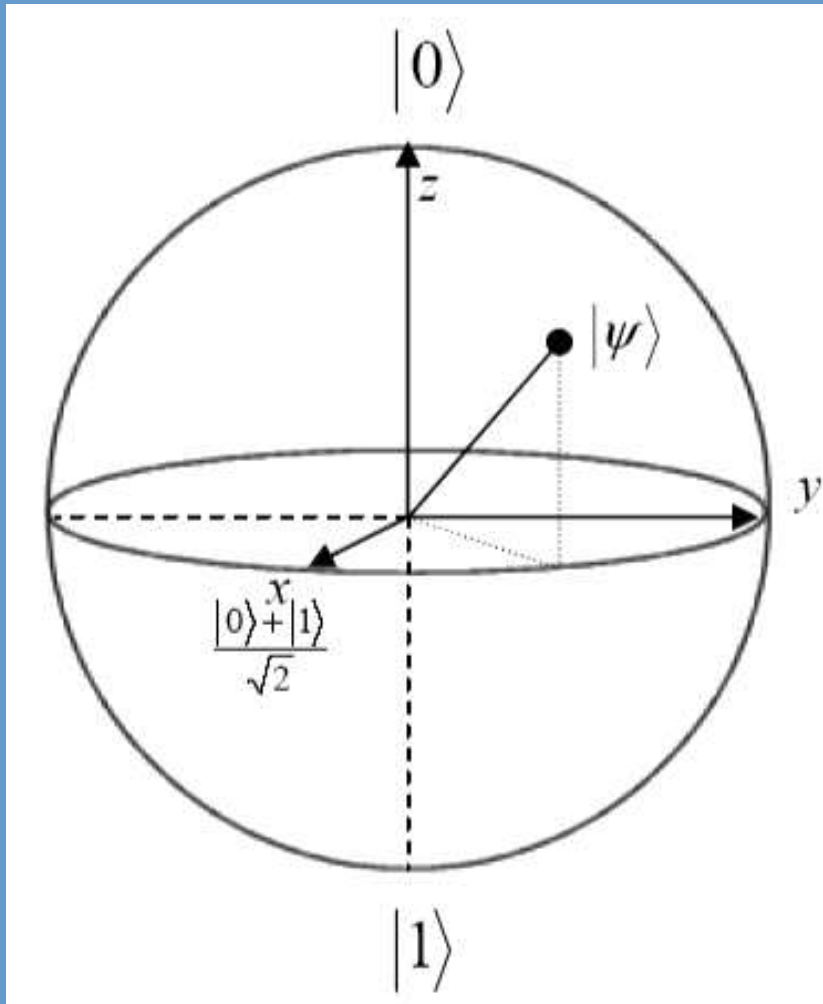
- The classical reversible circuits are a subset of Quantum circuits.
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- For example the (Quantum) Not gate:

$$x_0 \text{ ---} \boxed{\textit{Rot } u} \text{---} \neg x_0 \qquad u = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The Hadamard Transform





Irreversible Functions

- Not all functions are logically reversible.

$$\textit{And} \in (\textit{Bool}, \textit{Bool}) \rightarrow \textit{Bool}$$

$$\textit{And} (\textit{True}, \textit{True}) = \textit{True}$$

$$\textit{And} (a, b) = \textit{False}$$



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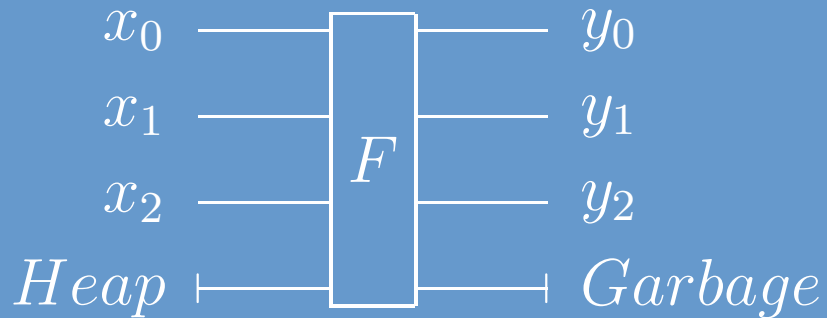
$$\text{ReverseAnd} \text{ False} = ?$$

- Using **Heap** inputs and **Garbage** outputs we can create reversible circuits from irreversible functions.



Heap and Garbage

- An example Circuit:





Heap and Garbage

- The **Heap** is extra input bits (or qubits) that have been set to 0 .



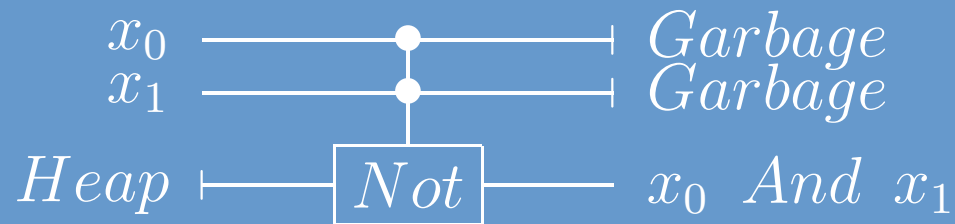
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- The **Heap** is extra input bits (or qubits) that have been set to 0 .
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Heap and Garbage

- The **Heap** is extra input bits (or qubits) that have been set to 0 .
- The **Garbage** contains any extra information that would normally be lost by an irreversible function.
- The reversible circuit for the *And* function is: (The Toffoli gate)





Reversible "And"

- The equivalent function for the reversible **And** circuit is:

$$\text{And} \in (\text{Bool}, \text{Bool}, \text{False}) \rightarrow (\text{Bool}, \text{Bool}, \text{Bool})$$

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- and Reversed as:

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What Now?



- What can we do with Heap and Garbage?



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- What can we do with Heap and Garbage?
- Laws for reversible circuits with Heap and Garbage.
- That hold Classically...
- and in the Quantum case.



The 3 Laws - The First Law

- Garbage Collection



The 3 Laws - The Second Law



- The Uselessness of Garbage Processing.



The 3 Laws - The Third Law



- The Law of Useless Heap Processing
if $h \vec{0} = \vec{0}$ then





Quantum or Classical

- What are the differences for Classical and Quantum cases?



Quantum or Classical

- What are the differences for Classical and Quantum cases?
- Measurement and Decoherence.



Quantum or Classical

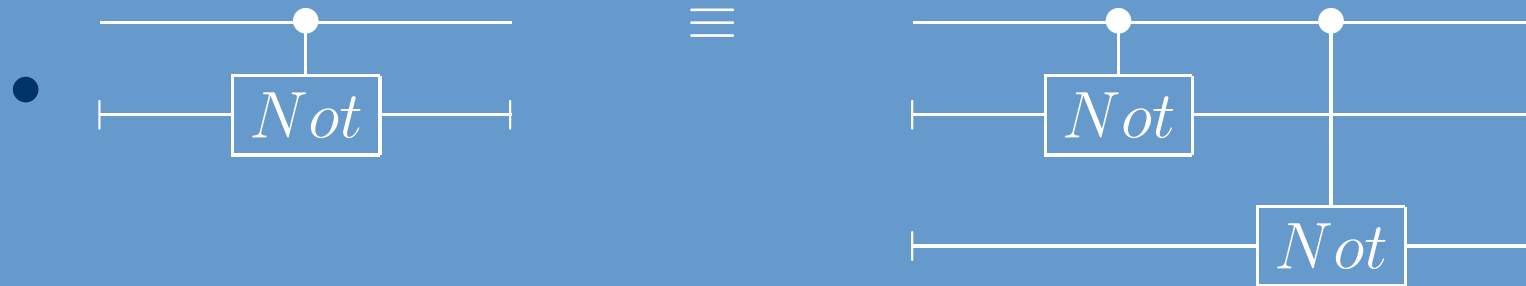
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Quantum or Classical

- What are the differences for Classical and Quantum cases?
- Measurement and Decoherence.
- The Classical subset of Quantum.
- The Measurement Postulate.

The Measurement Postulate



Measurement Postulate Proof



- The first step of the proof is to show that the following two circuits are equivalent.



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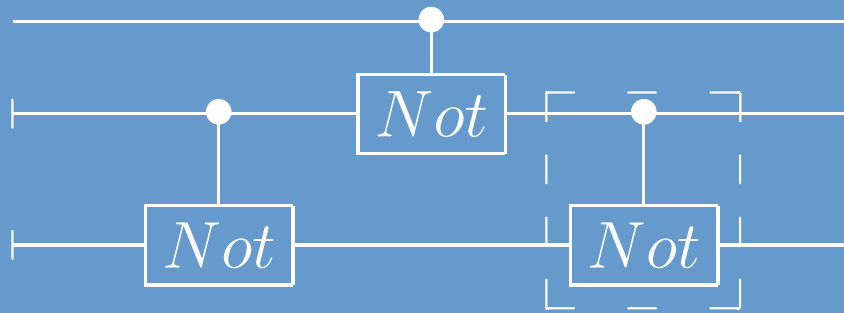


- These circuits are only made from elements in the Classical subset of the Quantum circuits

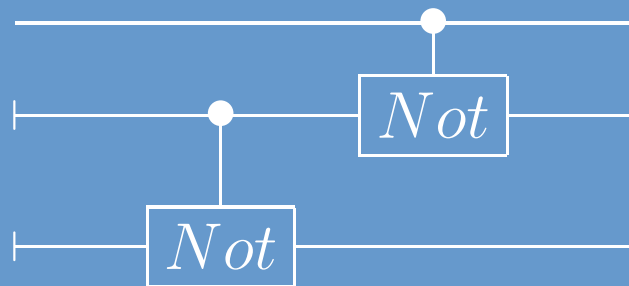
Measurement Postulate Proof



- Now we can use the Second law to eliminate the third controlled not.



≡



Measurement Postulate Proof



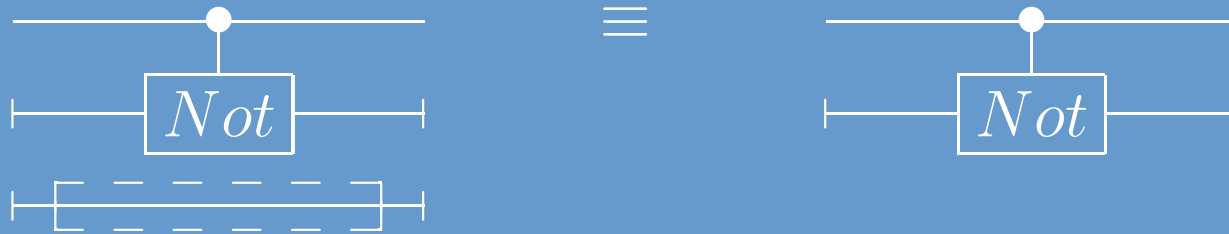
- The controlled not preserves zeros, so we can use the Third law to eliminate the first controlled not.



Measurement Postulate Proof



- Finally we remove the bottom wire by means of the First law.



Summary



- Reversible Circuits



Summary

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- Quantum Circuits



Summary

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- Irreversible Functions



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Summary

- Reversible Circuits
- Quantum Circuits
- Irreversible Functions
- Heap and Garbage
- The Three Laws
- The Measurement Postulate
- So what is next... ?



The End... ?

- Future work...
 - Completeness?
 - ...



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- Future work...
 - Completeness?
 - ...
- Thankyou for listening.



The End... ?

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- Any Questions? Ask me now, find me later, or email me.
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