

Comparing parallel and sequential Selfish Routing in the Atomic Players setting

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Outline

- 1 Introduction
 - The Price of Anarchy
 - Traffic Equilibrium Paradoxes
- 2 The model
 - Parallel setting
 - Sequential setting
 - Some results
 - Conclusions and Future works

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Introduction

Fundamental problem :
Efficient routing in large
traffic and communication
networks.

Problems:

- Many independent agents.
- Load dependant **latency**.

Solution: Selfish routing



Selfish routing

- Each user chooses a strategy to **minimize** his own cost, given the other players' strategies.
- Expect the routes chosen by users to form a **Nash equilibrium**
- **Nash equilibrium** : a strategies profile such that no user has incentive to change unilaterally his own strategy.

How efficient is the selfish routing?

- Equilibria of noncooperative game are typically **inefficient**.

Prisoner's dilemma

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		Deny	Confess
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	Confess	0, -9	-6, -6

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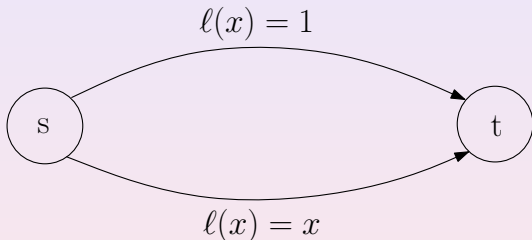
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How efficient is the Selfish Routing? (Cont.)

Example : Route a one unit flow from s to t

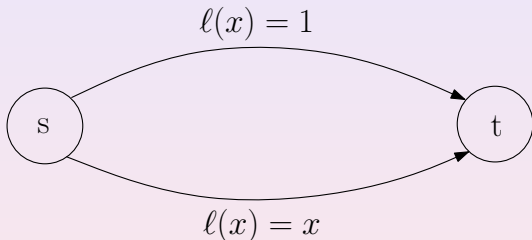


Question : what will selfish network users do?

- assume that every user wants to have a **smallest-possible delay**.

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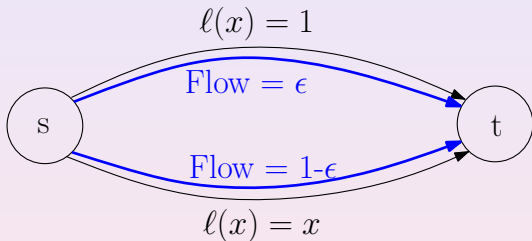


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Solution : all flows will take the bottom link.

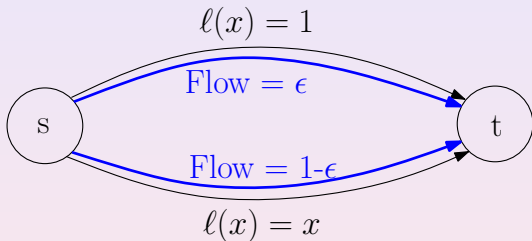


Because :

- If $\epsilon > 0$, the delay experienced by the flow on the bottom link is < 1 .
- If $\epsilon = 0$, no one has incentive to move.
- All flows experience a delay of 1.

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Is that the optimal solution?

Solution : the flow is splitted equally.

Delay :

- The top half has 1 unit of delay.
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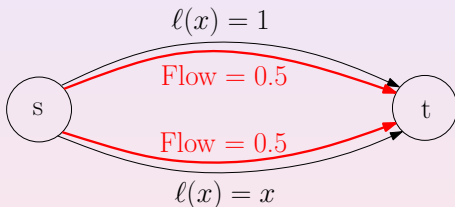
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The Price of Anarchy

Price of anarchy [Papadimitriou 2001] – “competitive analysis for noncooperative games”

Definition :

$$\text{POA} = \frac{\text{Worst case Nash equilibrium}}{\text{Optimal solution}}$$

Price of anarchy measures the “price” of not having central coordination in system.

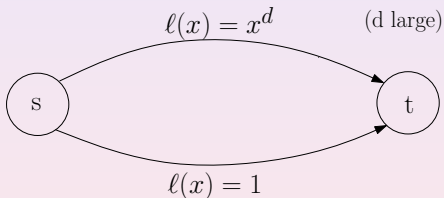
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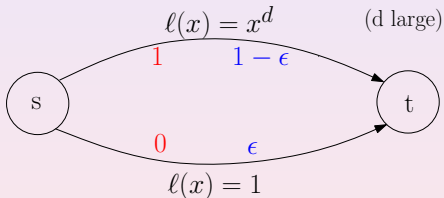
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Social cost :

- **Nash:** $1 \cdot 1^d + 0 \cdot 1 = 1$
- **Optimal:** $\epsilon \cdot 1 + (1 - \epsilon) \cdot \epsilon^d \rightarrow 0$ when ϵ close to 0

General latency function(cont.)

Different approach :

Bicriteria Results[Roughgarden/Tardos 2000] : for every network,

the cost of Nash flow of traffic rate $r \leq$ the cost of minimum flow of traffic rate $2r$.

Linear latency function

Theorem [Roughgarden/Tardos 2000] :

If latency function is of the form : $l_e(x) = a_e x + b_e$
where a_e and $b_e \geq 0$,

the cost of a Nash flow is at most $4/3$ times that of the
minimum-latency flow.

Traffic Equilibrium Paradoxes

Intuitively, less players \implies smaller social cost

- does not hold in general
- Numerical example [Catoni/Pallottino 1991] :
- Route a flow of 630 from s_1 to t_1 and another 630 from s_2 to t_2 .

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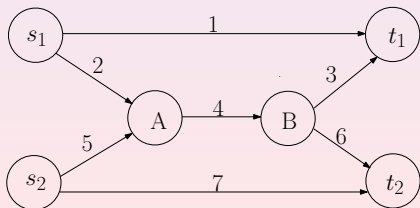
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$$l_1 = f_1 + 30;$$

$$l_4 = f_4 + 60;$$

$$l_7 = f_7$$

$$l_2 = l_3 = l_5 = l_6 = 0$$

Numerical example

- Consider four models :
 - **User Equilibrium** – a large population of very small network users
 - **System Equilibrium** – a single user
 - **Mixed Behavior Equilibrium** – one CN-player with many very small network users
 - ⇒ **CN-player** – a player controls an amount of splittable flow
 - **Cournot-Nash Equilibrium** – two CN-players

- Number of players : $UE > ME > CNE > SE$

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Traffic Equilibrium Paradoxes(Cont.)

	UE	ME	CNE	SE
$f(1, 1)$	420	360	382	420
$f(1, 4)$	210	270	248	210
$f(2, 4)$	180	150	238	195
$f(2, 7)$	450	480	392	435
$C(1)$	283500	270000	292792	286650
$C(2)$	283500	302400	283612	279900
Social cost	567000	572400	576404	566550

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The Model

Mathematical Model

- A single commodity network congestion game.
- 2 parallel edges.
- m CN-players, each controls one unit of splittable flow.
- A linear latency function $\ell_e(x) = a_e x + b_e$.
- Cost experienced by player i : $c_i = \sum_{e \in E} \ell_e(f_e) \cdot f(i, e)$.

We compare the social cost of two models:

- Parallel setting
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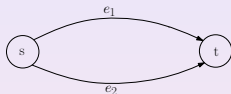
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Parallel setting

- Players make decision **simultaneously**.
- Expect players to form a **Nash equilibrium**.



Property 1 : At Nash equilibrium, every player splits his flow in the same proportion.

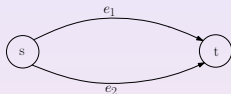
Property 2 : If latency functions are equal, every player splits his flow equally among edges.

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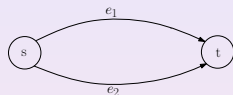
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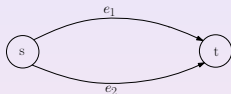
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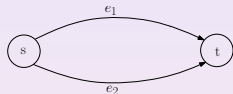
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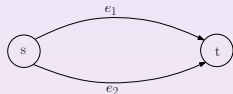
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- **Perfect information** – user is perfectly informed of all the event that have previously occurred.
- We solve the game using **backward induction**.



- stage m : player m solves $\min c_m(a_1, a_2, \dots, a_m)$ given the actions a_1, \dots, a_{m-1} previously chosen. Denote the solution by $R_m(a_1, \dots, a_{m-1})$.
- stage $m-1$: player $m-1$ solves $\min c_{m-1}(a_1, a_2, \dots, a_{m-1}, R_m(a_1, \dots, a_{m-1}))$

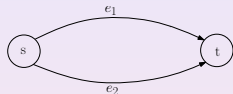
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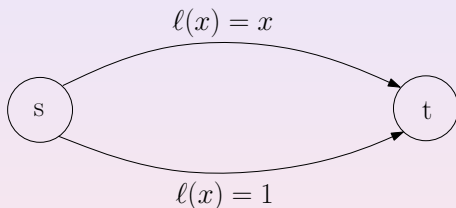
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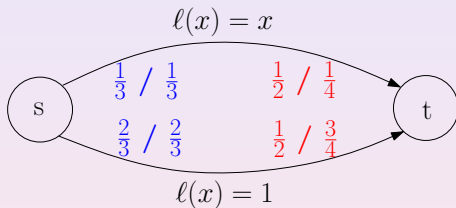


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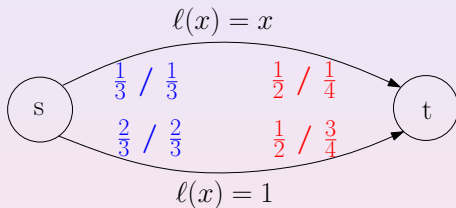
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Some results

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- Properties for parallel setting and sequential setting.
- Bounds on the ratio.

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