

Fixed-Point-Free is NP-complete

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The problem FPF

- Ω : a finite set $\{1, 2, \dots, n\}$
- G : a permutation group on Ω , i.e, a subgroup of $\text{sym}(\Omega)$ (or S_n). Assume G is given via a set of generators.
- the natural action $\Omega \times G \rightarrow \Omega$.
- an element $g \in G$ is called fixed point free (derangement) if $\text{fix}_\Omega(g) = \{\alpha \in \Omega \mid \alpha g = \alpha\} = \emptyset$.
- all such elements form a subset of G , denoted by:
 $\text{FPF}(G) = \{g \in G \mid \text{fix}_\Omega(g) = \emptyset\}$.
- Question: Given any G acting on the set Ω , is $\text{FPF}(G) = \emptyset$?

Example

$$\Omega = \{1, 2, \dots, 6\}$$

● $\text{FPF}(G) \neq \emptyset$:

● Trivial Case: Transitive action (orbit number is 1) via orbit-counting lemma.

● Nontrivial Case:

$$G = \{id, (1\ 2\ 3), (4\ 5\ 6), (1\ 2\ 3)(4\ 5\ 6)\}.$$

● $\text{FPF}(G) = \emptyset$:

● Trivial case: some orbit is of size 1;

● Nontrivial case:

$$G = \{id, (1\ 2)(3\ 4), (1\ 2)(5\ 6), (3\ 4)(5\ 6)\}.$$

An NP-complete problem: NAESAT

- **NP-complete problem: 3-SAT**
 - a finite set of boolean variables U ;
 - a collection of clause $C = \{c_1, \dots, c_m\}$, each has the form $c_k = a_{k,1} \vee a_{k,2} \vee a_{k,3}$ where $a_{k,i} = u_i$ or \bar{u}_i .
 - Question: Is there a satisfying truth assignment for all clauses.
 - Example: $U = \{u_1, u_2, u_3\}$; $c_1 = u_1 \vee u_2 \vee u_3$,
 $c_2 = \bar{u}_1 \vee u_2 \vee u_3$.
- **NAESAT**: In no clauses are all three literals equal in truth value.
- **NAESAT is NP-complete.**

Reduction 1: Gadgets

- The Variable Gadget: the cycle $t_i = (2i - 1, 2i)$ for each variable u_i .
- The Clause Gadget: Assume $c_k = a_{k,1} \vee a_{k,2} \vee a_{k,3}$.
 - $d_{k,1} = (d + 1, d + 2)(d + 3, d + 4)$ to the element $a_{k,1}$
 - $d_{k,2} = (d + 1, d + 3)(d + 2, d + 4)$ to the element $a_{k,2}$
 - $d_{k,3} = (d + 1, d + 4)(d + 2, d + 3)$ to the element $a_{k,3}$where $d = 2n + 4(k - 1)$.

Reduction 2: Construction

- $G = \langle g_1, g'_1, \dots, g_n, g'_n \rangle$ Where the cycles for each generator is given as follows:
 - t_i to g_i, g'_i .
 - $d_{k,j}$ to g_i if $a_{k,j} = u_i$
 - $d_{k,j}$ to g'_i if $a_{k,j} = \bar{u}_i$
- For an instance of **NAESAT** with n variables and m clauses, we have $|\Omega| = 2n + 4m$, and G having $2n$ generators. This means the reduction is polynomial.
- $\text{FPF}(G) \neq \emptyset$ iff the corresponding **NAESAT** is satisfiable.

Example

- **NAESAT:** $U = \{u_1, u_2, u_3\}$; $c_1 = u_1 \vee u_2 \vee u_3$,
 $c_2 = \bar{u}_1 \vee u_2 \vee u_3$.
- **FPF:** $G = \langle g_1, g'_1, g_2, g'_2, g_3, g'_3 \rangle$ where
 - $g_1 = (1\ 2)(7\ 8)(9\ 10)$
 - $g'_1 = (1\ 2)(11\ 12)(13\ 14)$
 - $g_2 = (3\ 4)(7\ 9)(8\ 10)(11\ 13)(12\ 14)$
 - $g'_2 = (3\ 4)$
 - $g_3 = (5\ 6)(7\ 10)(8\ 9)(11\ 14)(12\ 13)$
 - $g'_3 = (5\ 6)$
- satisfying assignments: $\{(T, T, F), (T, F, T), (F, T, F), (F, F, T)\}$
 - $\text{FPF}(G) = \{g_1 g_2 g'_3, g_1 g'_2 g_3, g'_1 g_2 g'_3, g'_1 g'_2 g_3\}$
 - The reduction is 1-1.

Metrics on Permutations

- Def: Hamming Distance $H(\pi, \sigma) = \#\{i : \pi(i) \neq \sigma(i)\}$.
- Def: Hamming Weight $W_H(\pi) = H(\pi, Id)$.
- Fact: $FPF(G) = \{\pi : |W_H(\pi)| = n\}$.
- Many other distances: Cayley, Ulam, Footrule, Kendall's Tau, Spearman's rank correlation.....
- Given a distance on permutation group, the corresponding Max(Min) Weight problem is to decide the Max.(Min.) weight.

Conclusions

- FPF is NP-complete. (independently by C.Buchheim & M.Jünger)
- #FPF is #P-complete
- Max(Min)-Hamming weight problem is NP-hard. (also B & J)
- The corresponding Max(Min)-weight problem is:
 - NP-hard: Cayley, Ulam, Footrule, Kendall's Tau, Spearman's rank correlation.....
 - P: $l_{\infty}(\pi, \sigma) = \max_{1 \leq i \leq n} |\pi(i) - \sigma(i)|$
- To decide the nontrivial Max(Min) Cycles is NP-hard.
- To decide the permutation character is NP-hard.

Open problems

- Characterize respectively the distance that lies in NP-hard and P.
- The complexity of counting $|FPF(G)|$ when G is transitive, note now $FPF(G) \neq \emptyset$ by orbit counting lemma.
- Approximately count $|FPF(G)|$.
- Compute the irreducible character.