Quantified Probability Logics: Expressibility vs. Computability

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Let $\mathscr{X} = \{x_i \mid i \in \mathbb{N}\}$ be the set of *variables*, $C = \{c_i \mid i \in I\}$ the set of *constants*, where I is a non-empty computable subset of $\mathbb{N}$. Define the collection of *e-terms* to be the smallest set containing $\mathscr{X} \cup C$ and such that if t_1 and t_2 are e-terms, then $\overline{t_1}$ and $t_1 \cap t_2$ are e-terms as well. By a $\mathscr{L}_\mu^C$*-atom* we mean an expression of the sort

$$f(\mu(t_1), \ldots, \mu(t_n)) \leqslant g(\mu(t_{n+1}), \ldots, \mu(t_{n+m})),$$

where f and g are polynomials with coefficients in $\mathbb{Q}$, and $t_1, \ldots, t_{n+m}$ are e-terms. Then $\mathscr{L}_\mu^C$-*formulas* are obtained from $\mathscr{L}_\mu^C$-atoms by closing under $\neg$, $\wedge$ and applications of $\forall x$, with x in $\mathscr{X}$; we abbreviate $\neg\forall x \neg\Phi$ as $\exists x \Phi$, as usual.

All the languages under consideration are interpreted over the class of *discrete probability spaces* (cf. [1]), each of which is represented by a triple $\langle \Omega, \mathscr{A}, \mathsf{P} \rangle$, where Ω is a countable (i. e., finite or countably infinite) set of *possible worlds*, $\mathscr{A}$ is a σ-algebra of subsets of Ω, and P is a discrete probability measure on $\mathscr{A}$ (extending some discrete distribution over Ω). Suppose that $\mathfrak{A} = \langle \Omega, \mathscr{A}, \mathsf{P} \rangle$ is a space of this type. The semantics of the quantifier-free fragment is, in fact, standard: for a quantifier-free $\mathscr{L}_\mu^C$-formula Φ and a valuation $v : \mathscr{X} \cup C \to \mathscr{A}$, define

$$\mathfrak{A} \Vdash \Phi[v]$$

by interpreting variables/constants as their images under v, $\overline{t_1}$ as the complement of t_1, $t_1 \cap t_2$ as the intersection of t_1 and t_2, and μ as P (the resulting expression is then verified in the ordered field of reals). The above semantics can be further expanded to arbitrary $\mathscr{L}_\mu^C$-formulas if quantifiers are viewed as ranging over: i) all events of $\mathscr{A}$; ii) all events of $\mathscr{A}$ expressible by ground e-terms. Denote the *quantified probability logics* corresponding to Items (i) and (ii) as QPL^C and $\mathsf{QPL}_\circ^C$, respectively. In case C is infinite, the $\mathsf{QPL}_\circ^C$-validity problem is known to be Π_1^1-complete [2] (and is easily shown to be decidable whenever C is finite). On the other hand, the QPL^C-validity problem is m-equivalent to the second-order theory of $\langle \mathbb{N}, +, \times \rangle$ (even with no constants). Also, a bunch of results on prefix fragments of $\mathsf{QPL}_\circ^C/\mathsf{QPL}^C$ have been established — those specifying the least such fragments for which the validity is undecidable.

The present talk is devoted to various aspects of the foregoing languages, focusing mainly on issues of expressibility and computability — and this, in turn, forces us to (re)examine certain topics in logic, as the experience in the area shows (e. g., the Π_∞^1-compexity for QPL^C can be proved using an alternative description of the analytical hierarchy from [3]). So the study of quantified probabilistic languages is intimately connected with integrating/improving different definability and decidability techniques.

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References

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