Computing Solution Operators of Boundary-value Problems for Some Linear Hyperbolic Systems of PDEs (Extended Abstract)

Svetlana Selivanova *
S.L. Sobolev Institute of Mathematics
Siberian Branch of the Russian Academy of Sciences
Novosibirsk, Russia
s_seliv@math.nsc.ru
and
Victor Selivanov†
A.P. Ershov Institute of Informatics Systems
Siberian Branch of the Russian Academy of Sciences
Novosibirsk, Russia
vseliv@iis.nsk.su

Abstract

We discuss possibilities of application of Numerical Analysis methods to proving computability, in the sense of the TTE approach, of solution operators of boundary-value problems for systems of PDEs. We prove computability of the solution operator for a symmetric hyperbolic system with computable real coefficients and dissipative boundary conditions, and of the Cauchy problem for the same system (in this case we also prove computable dependence on the coefficients) in a cube $Q \subseteq \mathbb{R}^m$. Such systems describe a wide variety of physical processes (e.g. elasticity, acoustics, Maxwell equations). Moreover, many boundary-value problems for the wave equation also can be reduced to this case, thus we partially answer a question raised in [WZ02]. Compared with most of other existing methods of proving computability for PDEs, this method does not require existence of explicit solution formulas and is thus applicable to a broader class of (systems of) equations.

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1 Introduction

We consider boundary-value problems for systems of PDEs of the form

$$\begin{cases} L\mathbf{u}(y) = f(y) \in C^p(\Omega, \mathbb{R}^n), \quad y \in \Omega \subset \mathbb{R}^k \\ \mathcal{L}\mathbf{u}(y)|_\Gamma = \varphi(y\mid_\Gamma) \in C^q(\Gamma, \mathbb{R}^n),\ \Gamma \subseteq \partial\Omega, \end{cases} \tag{1}$$

where L and $\mathcal{L}$ are differential operators (the differential order of $\mathcal{L}$ is less than the one of L), Γ is a part of the boundary $\partial\Omega$ of some area Ω. In particular, if $\Gamma = \{t = 0\}$ and t is among the variables $y_1, y_2, \dots, y_k$, then (1) is a Cauchy (or initial-value) problem. Assuming existence and uniqueness of the solution $\mathbf{u}$ in Ω, we study computability properties of the solution operator $R : (L, \mathcal{L}, f, \varphi) \mapsto \mathbf{u}$. Note that in (1) the number k of "space" variables $y_1, y_2, \dots, y_k$ is not necessarily equal to the number n of the unknown functions $u_1, u_2, \dots, u_n$, e.g. for the linear elasticity equations (6) we have $n = 9, k = 4$.

Computability will be understood in the sense of Weihrauch's TTE approach [We00]. Recently the following main achievements in the field of study of computability properties of PDEs were made. Computability of solution operators of initial-value problems for the wave equation [WZ02], Korteveg de Vries equation [GZZ01, WZ05], linear and nonlinear Schrödinger equations [WZ06] was established; also computability of fundamental solutions of PDEs with constant coefficients $Pu = \sum\limits_{|\alpha|\leq M} c_\alpha D^\alpha u = f$ was proved in [WZ06-2]. Most of the methods of the mentioned papers are based on a close examination of explicit solution formulas and the Fourier transformation method, except for the paper [WZ05] where a method based on fixed point iterations is introduced. In these papers, the initial data and solutions are mainly assumed to belong to some Sobolev classes of generalized functions.

As is well-known, explicit solution formulas for boundary-value problems (even for the Cauchy initial-value problems) exist rarely. Even for the simplest example of the wave equation the computability of the solution operator for boundary-value problem was formulated in [WZ02] as an open question, and we have not seen any paper where this question would be answered. Results of our paper provide, in particular, a positive answer to this question for the case of computable coefficients and dissipative boundary conditions.

In [SS09] we propounded an approach to study computability properties of PDEs based on finite-dimensional approximations (difference schemes widely used in Numerical Analysis) and established computability, in the sense of the TTE approach, of the solution operator $\varphi \mapsto \mathbf{u}$ of the Cauchy problem for a symmetric hyperbolic system, with a zero right-hand part, of the form

$$\begin{cases} A\frac{\partial \mathbf{u}}{\partial t} + \sum\limits_{i=1}^{m} B_i \frac{\partial \mathbf{u}}{\partial x_i} = 0,\ t \geq 0, \\ \mathbf{u}|_{t=0} = \varphi(x_1, \dots, x_m). \end{cases} \tag{2}$$

Here $A = A^* > 0$ and $B_i = B_i^*$ are constant symmetric computable $n \times n$-matrices, $t \geq 0$, $x = (x_1, \ldots, x_m) \in Q = [0,1]^m$, $\varphi : Q \to \mathbb{R}^n$ and $\mathbf{u} : Q \times [0, +\infty) \rightharpoonup \mathbb{R}^n$ is a partial function acting on the domain H of existence and uniqueness of the Cauchy problem (2). In [SS09] computability of the domain H (which is a convex polyhedron depending only on A, B_i) was also proved. The operator R mapping a C^{p+1} function φ to the unique C^p solution ($p \geq 2$) is computable, if the norms of the first and second partial derivatives of φ are uniformly bounded.

Such systems can be used to describe a wide variety of physical processes like those considered in the theories of elasticity, acoustics, electromagnetism etc., see e.g. [Fr54, God71, God76, LL86, LL04, KPS01]. They were first considered in 1954 by K.O. Friedrichs [Fr54]. He proved the existence theorem based on finite difference approximations, in contrast with the Schauder-Cauchy-Kovalevskaya method based on approximations by analytic functions and a careful study of infinite series. The notion of a hyperbolic system (applicable also to broader classes of systems) is due to I.G. Petrovskii [Pe37], see also the very interesting discussion on different notions of hyperbolicity and their motivations in [Fr54].

Recall that a linear first-order differential operator $E = \sum\limits_{\mu=1}^{m} A_\mu \frac{\partial}{\partial x_\mu} + B$, where A_μ, B are real $n \times n$ matrices, $\mu = 1, 2, \ldots, m$, is called *hyperbolic* in the sense of Petrovskii, if there is a $\xi^0 \in \mathbb{R}^m$ such that, for all $\xi \in \mathbb{R}^m$, the matrix

$$\sum_{\mu=1}^{m} \xi_\mu A_\mu - \lambda \sum_{\mu=1}^{m} \xi_\mu^0 A_\mu$$

has real eigenvalues λ. In particular, if all the matrices A_μ, $\mu = 1, 2, \ldots, m$ are symmetric and one of them is positive-definite, as in (2), then the operator E is obviously hyperbolic in this sense.

The Friedrichs' method has turned out to be interesting from the computational point of view because it yields algorithms for solving PDEs's in the exact sense of Computable Analysis which are based on methods really used in Numerical Analysis.

In this paper we prove computability for a broad class of boundary-value problems for (2), by using the difference approximations approach stemming from the work [Fr54] and developed in [GR62, God71, God76, KPS01] and others. Many details of our proofs are similar to those of the proof of the existence theorem for the linear hyperbolic systems in [God71, God76] but, since we refer to more rigorous approach of Computable Analysis we are forced to establish several additional estimates.

Our study intensively uses the well-known classical theorem of the theory of difference schemes stating that the approximation and stability properties of a difference scheme imply its convergence to the solution of the correspondent differential equation in a suitable grid norm uniformly on steps.

The proofs of this paper also rely on the well-known fact that the ordered field of algebraic real numbers and some extensions of this field are strongly constructivizable (this is closely related to the Tarski's quantifier elimination for real closed fields, see e.g. [Ta51, BPR06]) which implies computability of necessary spectral characteristics of symmetric matrices with algebraic real coefficients. This makes obvious computability of all steps in the iterative process induced by the difference scheme used in this paper. This trick also leads to the improvement (and corrections of some inaccuracies in the proof) of the main result in [SS09] to the result that the solution operator for the Cauchy problem (2) is computable not only on φ but also on the coefficients A, B_i (under some additional assumptions).

Thus, our proofs make use of results in several fields: PDEs, difference schemes, computable analysis, computable fields. Unfortunately, they do not yield practically feasible algorithms for solving the initial and boundary value problems for PDEs (a main reason is that the problem of quantifier elimination for real closed fields is computationally hard [BPR06]). Search for such feasible algorithms is a natural next step in the study of computability properties of PDEs.

In Section 2 we describe the considered problems and assumptions we need to prove the computability of solution operators. In Section 3 we formulate precisely the main results of the paper. We conclude in Section 4 by a short discussion of more general systems (1).

2 Statement of the boundary-value problem and examples

Besides of the Cauchy problem (2) we now consider the following boundary-value problem:

$$\begin{cases} A\frac{\partial \mathbf{u}}{\partial t} + \sum\limits_{i=1}^{m} B_i \frac{\partial \mathbf{u}}{\partial x_i} = f, \\ \mathbf{u}|_{t=0} = \varphi(x_1, \ldots, x_m), \\ \Phi_i^{(1)} \mathbf{u}(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_m, t) = 0, \\ \Phi_i^{(2)} \mathbf{u}(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_m, t) = 0, \\ i = 1, 2, \ldots, m, \end{cases} \tag{3}$$

where

$A = A^* > 0$ and $B_i = B_i^*$ are fixed computable symmetric $n \times n$-matrices;

$0 \leq t \leq T$ for a computable real T;

$x = (x_1, \ldots, x_m) \in Q = [0, 1]^m$;

$\varphi \in C^{p+1}(Q, \mathbb{R}^n)$, $f \in C^p(Q \times [0, T], \mathbb{R}^n)$, $p \geq 2$ (for simplicity we let in this paper $f = 0$);

the boundary coefficients $\Phi_i^{(1)}$, $\Phi_i^{(2)}$ are fixed computable matrices meeting the following conditions:

1) The number of rows of $\Phi_i^{(1)}$ (respectively, $\Phi_i^{(2)}$) is equal to the number of positive (respectively, negative) eigenvalues of the matrices $A^{-1}B_i$;

2) The boundary conditions are assumed to be dissipative which means that

$$(B_i \mathbf{u}, \mathbf{u}) \leq 0 \text{ for } x_i = 0, \quad (B_i \mathbf{u}, \mathbf{u}) \geq 0 \text{ for } x_i = 1, \quad i = 1, 2, \ldots, m. \tag{4}$$

Note that assumptions 1) regarding the dimensions of the matrices Φ_i are needed for proving existence of a solution $\mathbf{u} \in C^p(Q \times [0, T], \mathbb{R}^n)$ of (3), while the assumption (4) provides uniqueness of the solution [Fr54, God71, Ev98, Jo66]. Moreover, these assumptions are needed [God76] for proving stability of the difference scheme constructed below in Section 4, which is one of the main ingredients in the proof of computability results.

An example of dissipative boundary conditions for the system (3) are conservative boundary conditions, stating that the energy flow through the boundary is constant:

$$\oint_S ([\tau A + \sum_{i=1}^{m} \xi_i B_i]\mathbf{u}, \mathbf{u}) dS = \int_{Q\times[0,T]} 2(f, \mathbf{u}) d\Omega, \tag{5}$$

where $(\tau, \xi_1, \ldots, \xi_m)$ is the extrinsic normal vector for the surface $S = \partial(Q \times [0, T])$.

E.g. for the linear elasticity equations (which constitute a symmetric hyperbolic system with 9×9-matrices)

$$\begin{cases} \frac{1}{2\mu}\frac{\partial \sigma_{ij}}{\partial t} - \frac{\lambda}{2\mu(3\lambda+2\mu)}\delta_{ij}\frac{\partial(\sigma_{11}+\sigma_{22}+\sigma_{33})}{\partial t} - \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}) = 0, \quad i, j = 1, 2, 3, \\ \rho\frac{\partial u_i}{\partial t} - \frac{\partial \sigma_{ij}}{\partial x_j} = 0, \quad i = 1, 2, 3, \end{cases} \tag{6}$$

in particular, the following boundary equations are conservative

$$\begin{cases} x_1 = 0,\ x_2 = 1 : \ \sigma_{11} = \sigma_{12} = \sigma_{13} = 0, \\ y_1 = 0,\ y_2 = 1 : \ \sigma_{12} = \sigma_{22} = \sigma_{23} = 0, \\ z_1 = 0,\ z_2 = 1 : \ \sigma_{13} = \sigma_{23} = \sigma_{33} = 0, \end{cases}$$

which means that the tangent stresses at any boundary are zero. Indeed, the energy conservation law (5) takes the form

$$\oint_S [-2\xi(u\sigma_{11} + v\sigma_{12} + w\sigma_{13}) - 2\eta(u\sigma_{12} + v\sigma_{22} + w\sigma_{23}) - 2\zeta(u\sigma_{13} + v\sigma_{23} + w\sigma_{33})]dS = 0.$$

Here u_i are the velocities, σ_{ij} is the stresses tensor (a symmetric 3×3–matrix with six independent variables), ρ is the density and λ, μ are the Lame coefficients.

Interestingly, the boundary-value problem for the wave equation

$$\begin{cases} p_{tt} - c_0^2(p_{xx} + p_{yy} + p_{zz}) = 0, \quad (x, y, z) \in \Omega = [0, 1]^3, \\ p|_{t=0} = \varphi(x, y, z), \\ p_t|_{t=0} = \psi(x, y, z), \\ p|_{\partial\Omega} = 0, \end{cases} \tag{7}$$

can be reduced, in several ways, to a symmetric hyperbolic system (see e.g. [Gor79, Ev98, Jo66]), in particular to the three-dimensional acoustics equations

$$\begin{cases} \rho_0 \frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} = 0, \\ \rho_0 \frac{\partial v}{\partial t} + \frac{\partial p}{\partial y} = 0, \\ \rho_0 \frac{\partial w}{\partial t} + \frac{\partial p}{\partial z} = 0 \\ \frac{\partial p}{\partial t} + \rho_0 c_0^2 (\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}) = 0, \\ p|_{t=0} = \varphi(x,y,z), \\ u|_{t=0} = -\frac{1}{\rho_0 c_0^2} \int\limits_0^x \psi(\xi,y,z) d\xi, \\ v|_{t=0} = 0, \\ w|_{t=0} = 0, \\ p|_{\partial\Omega} = 0, \end{cases} \tag{8}$$

where u, v, w are the velocities, p is the pressure, ρ_0 is the density and c_0 is the speed constant.

Obviously, such reduction can be done effectively, since integration is a computable operation. Thus the methods of proving computability for symmetric hyperbolic systems can be also applied to prove computability for the wave equation. This gives a partial answer to the open question on the wave equation raised in [WZ02]: the boundary-value problem for the wave equation is computable provided that it is dissipative (i.e., the corresponding boundary-value problem for a symmetric hyperbolic system to which the wave equation is reduced, is dissipative) and c_0 is a computable real.

3 Computability of the solution operators

In this section we give precise formulations of our main results.

We will work with several functional spaces most of which are subsets of the set $C(\mathbb{R}^m, \mathbb{R}^n) \simeq C(\mathbb{R}^m, \mathbb{R})^n$ of integrable continuous functions $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ equipped with the L_2-norm. In particular, we deal with the space $C(Q, \mathbb{R}^n) \simeq C(Q, \mathbb{R})^n$ (resp. $C^k(Q, \mathbb{R}^n)$) of continuous (resp. k-time continuously differentiable) functions $\varphi : Q \to \mathbb{R}^n$ equipped with the L_2-norm

$$||\varphi||_{L_2} = \left(\int_Q |\varphi(x)|^2 dx) \right)^{\frac{1}{2}}, \ |\varphi(x)|^2 = \langle \varphi, \varphi \rangle = \sum_{i=1}^n \varphi_i^2(x).$$

We will also use the sup-norm

$$||\varphi||_s = \sup_{x \in Q} |\varphi(x)|, \ ||u||_s = \sup_{(x,t) \in Q \times [0,T]} |u(x,t)|$$

on $C(Q, \mathbb{R}^n)$ or $C(Q \times [0,T], \mathbb{R}^n)$ and the sL_2-norm

$$||u||_{sL_2} = \sup_{0 \le t_0 \le T} \sqrt{\int_Q |u(x,t_0)|^2 dx}$$

on $C(Q \times [0,T], \mathbb{R}^n)$ where $T > 0$. Whenever we want to emphasize the norm we use notation like $C_{L_2}(Q, \mathbb{R}^n)$, $C_s(Q, \mathbb{R}^n)$ or $C_{sL_2}(Q \times [0,T], \mathbb{R}^n)$. Note that all introduced spaces are computable metric spaces.

The first main result concerns computability of the boundary-value problem (3) posed in Section 2.

Theorem 1 *Let $T > 0$ be a computable real and $M_\varphi > 0$, $p \geq 2$ be integers. Let $A, B_1, \ldots, B_m$ be fixed computable symmetric matrices, such that $A = A^* > 0$, $B_i = B_i^*$. Let $\Phi_i^{(1)}$, $\Phi_i^{(2)}$ $(i = 1, 2, \ldots, m)$ be fixed computable rectangular real non-degenerate matrices, with their numbers of rows equal to the number of positive and negative eigenvalues of $A^{-1}B_i$, respectively, and such that inequalities (4) hold.*

If $\varphi \in C^{p+1}(Q)$ satisfies

$$||\frac{\partial \varphi}{\partial x_i}||_s \leq M_\varphi, \; ||\frac{\partial^2 \varphi}{\partial x_i \partial x_j}||_s \leq M_\varphi, \; i, j = 1, 2, \ldots, m, \tag{9}$$

and meets the boundary conditions, then the operator $R : \varphi \mapsto \mathbf{u}$ mapping the initial function to the unique solution $\mathbf{u} \in C^p(H, \mathbb{R}^n)$ of the boundary-value problem (3) is a computable partial function from $C_s(Q, \mathbb{R}^n)$ to $C_{sL_2}(Q \times [0,T], \mathbb{R}^n)$.

The second main result concerns the initial value problem (2). It improves the main result of [SS09].

Theorem 2 *Let $M_\varphi > 0, M_A > 0, p \geq 2$ be integers, let $i = 1, \ldots, m$, and let $n_A, n_1, \ldots, n_m$ be cardinalities of spectra of A and of the matrix pencils $A - \lambda B_1, \ldots, A - \lambda B_m$, respectively (i.e., n_i is the number of distinct roots of the characteristic polynomial $det(A - \lambda B_i)$). Then the operator*

$$(A, B_1, \ldots, B_m, n_A, n_1, \ldots, n_m, \varphi) \mapsto \mathbf{u}$$

sending any sequence $A, B_1, \ldots, B_m$ of symmetric real matrices with $A > 0$ such that the matrix pencils $A - \lambda B_i$ have no zero eigenvalues,

$$||A||_2, ||A^{-1}||_2, ||B_i||_2 \leq M_A, \quad \lambda_{\min}^{(i)} < 0 < \lambda_{\max}^{(i)}, \quad i = 1, 2, \ldots, m, \tag{10}$$

the sequence $n_A, n_1, \ldots, n_m$ of the corresponding cardinalities, and any function $\varphi \in C^{p+1}(Q, \mathbb{R}^n)$ satisfying the conditions (9), to the unique solution $\mathbf{u} \in C^p(H, \mathbb{R}^n)$ of (2) is a computable partial function from the space $S_+ \times S^m \times \mathbb{N}^{m+1} \times C_s(Q, \mathbb{R}^n)$ to $C_{sL_2}(H, \mathbb{R}^n)$.

In Theorem 2, $\lambda_{\min}^{(i)}, \lambda_{\max}^{(i)}$ are respectively the minimal and maximal eigenvalues of the matrix pencil $A - \lambda B_i$, $S \subseteq \mathbb{R}^{n \times n}$ is the space of symmetric $n \times n$ matrices equipped with the Euclidean norm, S_+ is the space of symmetric positively definite matrices with the Euclidean norm, and $H \subseteq \mathbb{R}^{m+1}$ is the domain of correctness of (2), i.e., the maximal set where, for any $p \geq 2$ and $\varphi \in C^{p+1}(Q, \mathbb{R}^n)$, there exists a unique solution $\mathbf{u} \in C^p(H, \mathbb{R}^n)$ of the initial value problem (2).

The set H is known to be (see e.g. [God71]) a nonempty intersection of the semi-spaces

$$t \geq 0, \; x_i - \lambda_{\max}^{(i)} t \geq 0, \; x_i - 1 - \lambda_{\min}^{(i)} t \leq 0, \; (i = 1, \ldots, m)$$

of $\mathbb{R}^{m+1}$. We are especially interested in the case when H is a compact subset of $Q \times [0, +\infty)$ (obviously, a sufficient condition for this to be true is $\lambda_{\min}^{(i)} < 0 < \lambda_{\max}^{(i)}$ for all $i = 1, \ldots, m$; this is often the case for natural physical systems). In [SS09] we observed that the domain H for the problem (2) is computable from $A, B_1, \ldots, B_m$ (more exactly, the vector $(\lambda_{\max}^{(1)}, \ldots, \lambda_{\max}^{(m)}, \lambda_{\min}^{(1)}, \ldots, \lambda_{\min}^{(m)})$ is computable from $A, B_1, \ldots, B_m$; this implies computability of H in the sense of Computable Analysis [We00]).

Since, for each $i = 1, \ldots, m$, $\lambda_{\max}^{(i)}$ is the maximal and $\lambda_{\min}^{(i)}$ is the minimal eigenvalue of the matrix pencil $\lambda A - B_i$, and maximum and minimum of a vector of reals are computable [We00], it suffices to show that a vector $(\lambda_1, \ldots, \lambda_n)$ consisting of all eigenvalues of a matrix pencil $\lambda A - B$ is computable from A, B. But $(\lambda_1, \ldots, \lambda_n)$ is a vector of all roots of the characteristic polynomial of $\lambda A - B$, hence it is computable [We00, BHW03].

Remark 1 *1. Besides the condition $\lambda_{\min}^{(i)} < 0 < \lambda_{\max}^{(i)}$ in Theorem 2, some alternative natural conditions may be assumed. E.g., for one equation $\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} = 0$ the domain of correctness may be the intersection of semi-planes $\{t \geq 0\}$, $\{x \leq t\}$, $\{x \leq 1 + t\}$, and we search the solution in the intersection of semi-planes $\{t \geq 0\}$, $\{x \leq t\}$, $\{x = 1\}$. Our proof may be adjusted to similar modifications in a straightforward way.*

2. Note that Theorem 2 states the computability on coefficients A, B_i while Theorem 1 does not. The reason is that our proof of Theorem 2 (where we take rational fast Cauchy approximations to A, B_i) can not be straightforwardly adjusted to that of Theorem 1 because the dissipativity conditions in the last theorem might hold for the given matrix but not hold for the approximate matrices. Currently we do not know whether Theorem 1 may be strengthened to include computability on the real coefficients A, B_i.

3. In [SS09], we established a weaker result that the solution is computable provided that $A, B_1, \ldots, B_m$ are fixed computable matrices (in this case one can of course omit the conditions on spectra of A and of the matrix pencils). In the stronger formulation above, the proof requires additional considerations.

4 Conclusion

As noted in Section 2, the wave equation (7) can be reduced (not in a unique way) to a symmetric hyperbolic system, thus its solution operator is computable which gives an affirmative answer to the question of paper [WZ02], for the case of dissipative boundary conditions and an initial function with uniformly bounded derivatives. Note, however, that we consider not Sobolev H^s spaces of generalized functions, but C^k spaces of continuously differentiable functions; the smoothness k, which is to be assumed, depends on the particular problem under consideration. To prove computability of the generalized solutions it would be probably suitable to use finite element methods.

The restriction on the initial function seems to be rather strong from Computable Analysis viewpoint, but it is very natural for Numerical Analysis (though we have never seen it explicitly in formulations of Numerical Analysis theorems). Indeed, it is well known, that any initial (and right-hand part) function can be represented in the form of a Fourier series (consider for simplicity the one-dimensional case) $\varphi(x) = \sum_n a_n e^{inx}$, the differentiation of which gives $\sum_n na_n e^{inx}$, i.e.“fast oscillating” functions lead to large derivatives and hence large convergence constants which can make the scheme not convergent on a real computer.

A similar situation is with the additional assumptions of Theorem 2 about the apriori knowledge of the spectra of A and of matrix pencils and the absence of zero eigenvalues: the assumptions correspond well to the experience of numerical analysts. Namely, violation of these assumptions may lead to computational instabilities. We currently do not know whether our results hold without these assumptions. Note, however, that the assumptions are fulfilled (for physical reasons) for some important systems invariant under rotations [GM98].

Finally, we would like to point out that it would be interesting to study the computational complexity of the considered problems, in the spirit of [Ko91]. The algorithms suggested in our paper are very time- and space-consuming because they intensively use exact computations with algebraic (and even with some non-algebraic computable) reals which are known to be very hard. Currently we do not know any feasible algorithm to compute the solution operators studied in this paper.

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