A Theorem on Computable Martingales

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Many results from algorithmic randomness theory suggest that an infinite binary sequence is far from being random if and only if it has few derandomization power. In this talk, we add new evidence supporting this view.

A *dividend* is a function $\delta : 2^{<\omega} \to [1,\infty)$. A *$\delta$-martingale* is a function $d : 2^{<\omega} \to [0,\infty)$ with $d(\sigma) = d(\sigma 0)/\delta(\sigma 0) + d(\sigma 1)/\delta(\sigma 1)$. The value $d(\sigma)$ indicates a gambler's capital based on a dividend δ when the history of coin-tosses is σ. If δ is a constant function 2, we simply say that d is a martingale. At each time n, if a gambler bets all her/his money on $x(n)$, her/his capital $d(x \upharpoonright n)$ along x would increase to $\prod_n \delta(x \upharpoonright n)$ dollars. A dividend δ is *legal* if $\prod_n \delta(x \upharpoonright n)$ diverges to infinity for any $x \in 2^\omega$. A δ-martingale d *frequently succeeds on* x if there exists a computable function $l : \omega \to \omega$ such that for every $n \in \omega$, $d(x \upharpoonright k) \geq n$ for some $k \in [l(n), l(n+1))$.

It is well-known that the probability of making much money as she/he wants is 0%. We may find out more about such events by seeing the internal structure of the σ-ideal of null subsets of Cantor space. Such σ-ideals have been deeply studied in modern set theory. Among them are the *strong measure zero* sets, the *meager-additive* sets, and the *null-additive* sets.

By borrowing a technique used in Pawlikowski's characterization of strong measure zero in set theory, we show the following equivalence for computable martingales.

Theorem. *The following are equivalent for an infinite binary sequence $x \in 2^\omega$:*

(i) *For every computable legal dividend δ, a computable δ-martingale frequently succeeds on x.*

(ii) *For every infinite binary sequence $y \in 2^\omega$, a computable martingale frequently succeeds on y if and only if an x-truth-table martingale frequently succeeds on y.*

(iii) *For every infinite binary sequence $y \in 2^\omega$, a computable martingale frequently succeeds on y if and only if a computable martingale frequently succeeds on $x + y$.*

Here, for infinite binary sequences $x, y \in 2^\omega$, the bitwise-addition $x+y$ is defined by $(x+y)(n) \equiv x(n) + y(n) \mod 2$.

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