# Closed Sets and Operators thereon: Representations, Computability and Complexity

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In this talk will see four (of many) seemingly different ways to encode closed/compact/convex/regular sets via infinite binary strings, how to feed them to an OTM, and how to compute with them set operators like union and intersection, point finding and projection, or even the image and the inverse of a given function. We will see how to measure the (second–order, parameterized; see [7]) complexity of these operations (and operators), and whether (or not) these set encodings (*representations*) are (polytime) equivalent.

**Background**: In *Computable Analysis*, not only encodings of real numbers, functions and operators are considered, but also representations of subsets of, say, $\mathbb{R}^d$ for fixed dimension $d$. Although, in theory, the characteristic function $\chi_S$ uniquely represents a set $S$, it is a discontinuous function and therefore (using TTE [2]) non-computable for non-trivial $S$ [2, Thm. 5.1.5]. However, using the topological properties of the metric space under consideration ($X$, for say), one *can* devise set-representations $f\colon \subseteq \{0,1\}^\omega \to \mathcal{A} := \{S \subseteq X \text{ closed}\}$:

- $\boldsymbol{\omega}_{\text{mem}}$ Asserts either $\overline{\text{ball}}(\boldsymbol{a}, r) \cap S \neq \emptyset$, or $\overline{\text{ball}}(\boldsymbol{a}, r) \not\subseteq S$ — [1, Def. 2.1.14] (this representation is strongly related to *convex optimization*; cf. [1, Cor. 4.3.12]);
- $\boldsymbol{\delta}_{\text{dist}}$ $[\rho^d_{\mathbb{R}} \to \rho_{\mathbb{R}\cup\{\infty\}}]$–representation of $S$' distance function $d_S$ — [2, Def. 5.1.6];
- $\boldsymbol{\kappa}_{\text{G}}$ Enumerates for each $n \in \mathbb{N}$ vectors $(\boldsymbol{a}_{n,i})_i \subset \mathbb{D}^d_n$ s. t. the *Hausdorff-distance* between $S$ and $\bigcup_i \overline{\text{ball}}(\boldsymbol{a}_{n,i}, 2^{-n})$ is $\leq 2^{-n}$ — [2, Def. 7.4.1(3)], [5, Def. 2.2].

In [6], Kawamura and Cook introduced *second-order representations* $\tilde{f} :\subseteq (\{0,1\}^* \to \{0,1\}^*) \to \mathcal{A}$ (such $\tilde{f}$ can be constructed from a 'classical' representation $f$). In addition to $\tilde{\omega}$, $\tilde{\psi}^{\text{dist}}$ and $\tilde{\kappa}_G$, consider:

- $\boldsymbol{\psi}_{\circledcirc}$ Asserts either $\overline{\text{ball}}(\boldsymbol{a}, 2r) \cap S \neq \emptyset$, or $\text{ball}(\boldsymbol{a}, r) \cap S = \emptyset$ — [6, Sec. 4.2.1].

This notion of *representation* helps to study mutual *second-order polytime translations* (i. e., (effective plus uniform) translations from one representation to another). Depending on the restriction (and additional parameters like dimension, inner points, radii), these representations are either *computably* (but *not polytime*) equivalent (cf. [2, Sec. 5], [3, Thm. 4.9]), different (cf. [4, Thm. 3.2.1]), or (and these are good news) *polytime equivalent* (even if one changes the norm underlying these set representations).

Having established the relations between representations, we study the (second-order, parameterized) *complexity* of *operators* like $\mathsf{Union}$, $\mathsf{Intersection}$, but also of $\mathsf{Image} : (f, S) \mapsto f[S]$, $\mathsf{Preimage}$ and $\mathsf{Inverse}$ (just to name some examples).

## References

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