Prefix-like Complexities and Computability in the Limit

Alexey Chernov Jürgen Schmidhuber

Istituto Dalle Molle di Studio sull'Intelligenza Artificiale, Lugano, Switzerland

Technische Universität München, Germany

Computability in Europe'06
University of Wales Swansea
July 2006

Limit Computation

Non-halting computation:

15 (computing...)

Limit Computation

Non-halting computation:

~~15~~ (computing...) 83 (computing...)

Limit Computation

Non-halting computation:

~~15~~	(computing...)	~~83~~	(computing...)
31	(computing...)		

Limit Computation

Non-halting computation:

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~~31~~	(computing...)	...	45 (computing...)

Limit Computation

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Limit computations $\approx$ computations with the oracle $0'$

Shoenfield Limit Lemma

Theorem

$f : \mathbb{N} \to \mathbb{N}$

$$f(n) = \lim_{m \to \infty} g(n, m), \;\; g(\cdot, \cdot) \textit{ is computable}$$

$$\Leftrightarrow \qquad f(\cdot) \textit{ is } 0'\textit{-computable}$$

Kolmogorov Complexity

$K(x)$ is the length of the minimal description of x

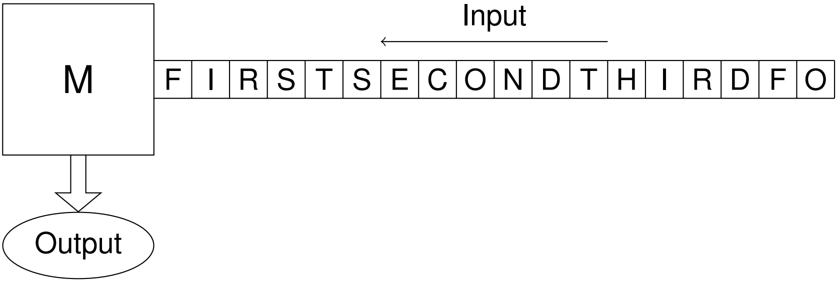
$$K_U(x) = \min \ell(y) \quad : \quad y \overset{U}{\mapsto} x$$

U is universal if $\forall V \, \exists C \quad \forall x \quad K_U(x) \leq K_V(x) + C$

A universal U exists:
$U(y) = U(\langle z, w \rangle) = U(\langle code(V), w \rangle) = V(w)$.

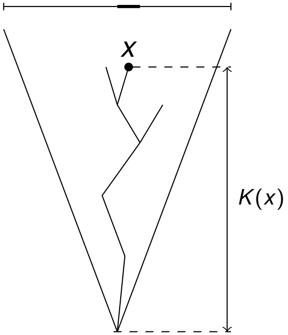
Prefix input

Machine M with sequential input (without delimiters)

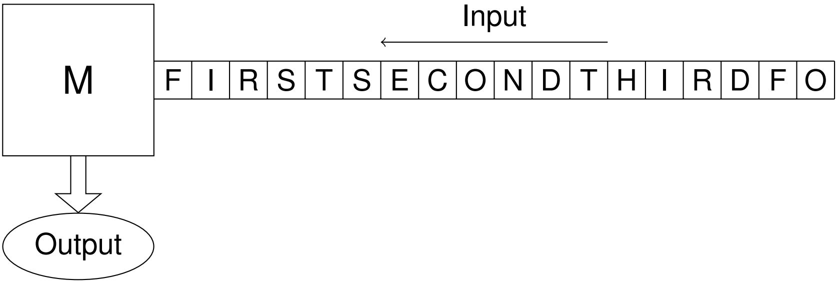


Prefix Complexities

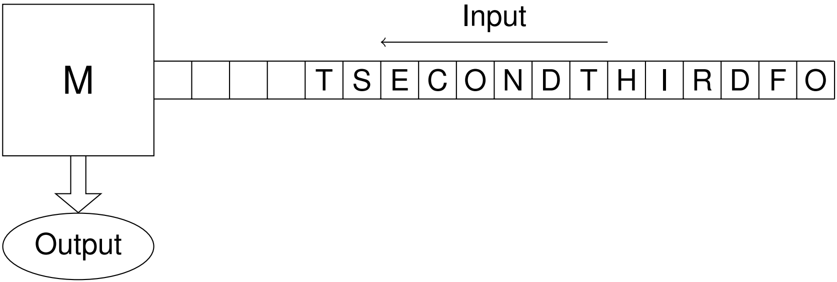
$$\min \ell(y)$$
$$U(y) = x$$



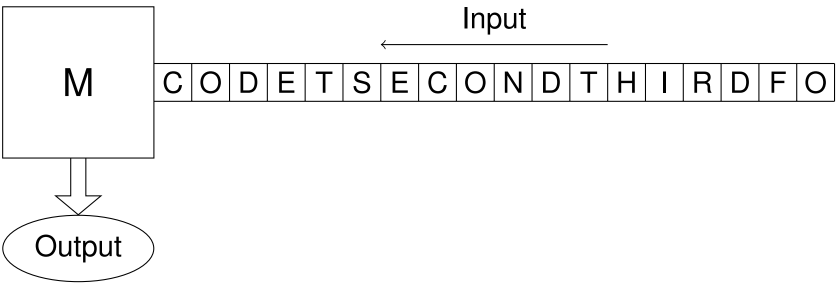
Weak prefix input



Weak prefix input



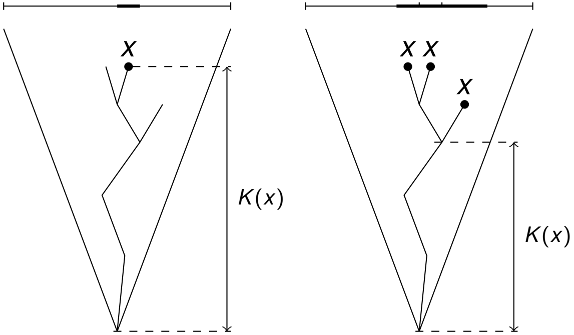
Weak prefix input



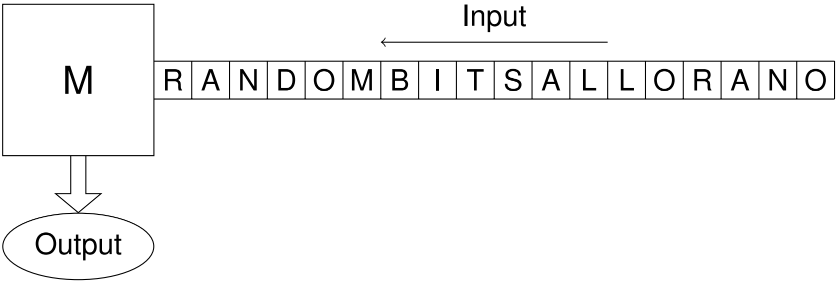
Prefix Complexities

$$\min \ell(y) \quad U(y) = x$$

$$\min \ell(y) \quad U(y\ldots) = x$$

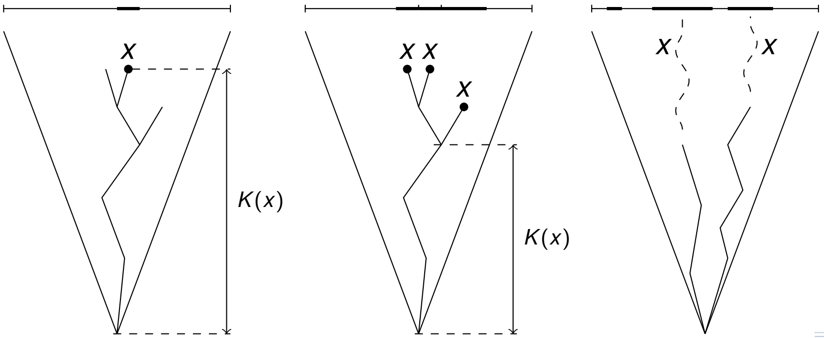


Random input

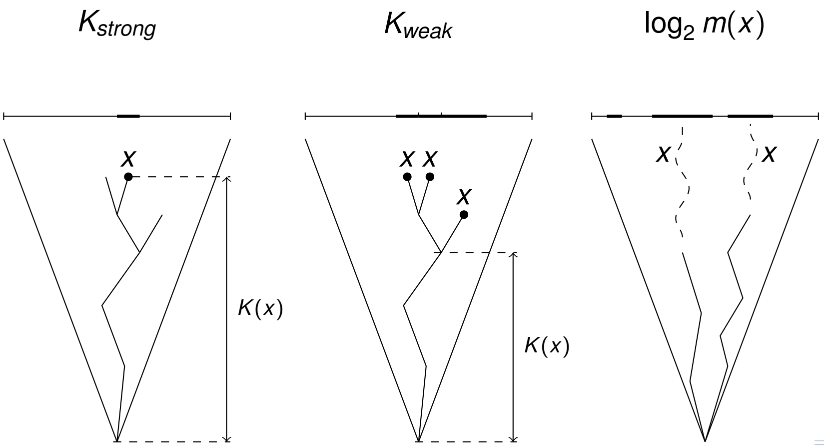


Prefix Complexities

$\min \ell(y)$	$\min \ell(y)$	$\log_2 \sum 2^{-\ell(y)}$
$U(y) = x$	$U(y\ldots) = x$	$U(y) = x$



Prefix Complexities



Coding Theorem

Theorem (Levin, 1974)

$K_{strong}(x) = \log_2 m(x) + O(1)$

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Corollary

$$K^{0'}_{strong}(x) = K^{0'}_{weak}(x) + O(1) = \log_2 m^{0'}(x) + O(1)$$

Limit and 0′-Complexities

Theorem

$$K^{lim}_{strong}(x) \neq K^{0'}_{strong}(x) + O(1)$$

$$K^{0'}_{strong}(x) \leq K^{lim}_{strong}(x) + O(1) \leq K^{0'}_{strong}(x) + O(\log K^{0'}_{strong}(x))$$

and the bounds are tight.

Limit and $0'$-Complexities

Theorem

$$K^{0'}_{weak}(x) = K^{lim}_{weak}(x) + O(1) = \log_2 m^{0'}(x) + O(1) = \log_2 m^{lim}(x) + O(1)$$

Prefix Complexities