Functional Interpretation and Modified Realizability Interpretation of the Double-Negation Shift

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Introduction

The purpose of the general program of proof mining is to develop techniques to extract additional information from proofs in mathematics and computer science.

Motivation and inspiration:

- Kreisel's program: G. Kreisel and others, several case studies, 1950s and onward.
- Analysing proofs in analysis: U.Kohlenbach and others, since 1990s, metatheorems and case studies.

Introduction

Main techniques of proof mining: Gödel's functional interpretation and Kreisel's modified realizability interpretation.

Analysing (ineffective) proofs in analysis (Peano arithmetic + dependent choice): Howard-Bezem majorizability $\Rightarrow$ monotone proof interpretations

Important step: computational interpretation of the double-negation shift (short: DNS) to interpret negative translation of dependent choice.

Introduction

- Spector(1962): Functional interpretation of DNS using so-called (Spector) bar-recursion, SBR (recursion over well-founded trees).
- Bezem(1985): Not only continuous functionals, but also strongly majorizable functionals are a model of (Spector) bar-recursion.
- Berardi, Bezem and Coquand(1998), later also Berger and Oliva(2002): modified realizability interpretation of double-negation shift using so-called modified bar-recursion, MBR.

Introduction

Aim of this talk:

- Investigate interpretations of finite versions of DNS and their respective prim. recursive interpretations.
- Understand intuition behind interpretations of DNS.
- Understand (and alleviate?) differences between functional interpretation and mr-interpretation of DNS (MBR is strictly stronger than SBR).
- Investigate if there is an mr-interpretation of DNS using only Spector's bar-recursion.

Overview

- Introduction
- **Functional Interpretation of DNS**
- Modified Realizability and DNS

Functional Interpretation of DNS

The double-negation shift, DNS, is the principle:

$$\forall x \neg\neg P(x) \to \neg\neg \forall x P(x),$$

for arbitrary formulas P.

For functional interpretation one needs to solve:

$$x = Y(C), \quad A(x,B) = C(Y(C)), \quad B(A(x,B)) = D(C),$$

i.e. give x, B, C in parameters Y, A, D.

Spector solved these equations using bar-recursion and with the condition $\forall C \exists n (Y(\overline{C,n}) < n)$.

Functional Interpretation of DNS

As finite version of the double-negation shift is

$$\forall k (\forall x \leq k \neg\neg P(x) \to \neg\neg \forall x \leq k P(x)).$$

This can be derived by induction from the intuitionistic principle

$$(\neg\neg P_0 \wedge \neg\neg P_1) \to \neg\neg(P_0 \wedge P_1)$$

and hence has a *primitive* recursive interpretation.

Functional Interpretation of DNS

What is the relationship between interpretations of this principle and interpretations of the full double-negation shift?

Interpretation of finite DNS leads to equations

$$A_0(B_0) = C_0, \qquad B_0(A_0(B_0)) = D_0(C_0, C_1),$$

$$A_1(B_1) = C_1, \qquad B_1(A_1(B_1)) = D_1(C_0, C_1),$$

which can be solved primitive recursively.

Functional Interpretation of DNS

From a proof of $(\neg\neg P_0 \wedge \neg\neg P_1) \to \neg\neg(P_0 \wedge P_1)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg\neg P_0 \wedge \neg\neg P_1^4}{\neg\neg P_1}\wedge E \qquad \dfrac{\dfrac{\dfrac{\dfrac{\neg(P_0 \wedge P_1)^3 \qquad \dfrac{P_0^1 \quad P_1^2}{P_0 \wedge P_1}\wedge I}{\bot}\to E}{\neg P_0}\to I(1) \qquad \dfrac{\neg\neg P_0 \wedge \neg\neg P_1^4}{\neg\neg P_0}\wedge E}{\bot}\to E}{\neg P_1}\to I(2)}{\bot}\to E}{\neg\neg(P_0 \wedge P_1)}\to I(3)}{(\neg\neg P_0 \wedge \neg\neg P_1) \to \neg\neg(P_0 \wedge P_1)}\to I(4)$$

one can extract a functional interpretation of finite DNS.

Functional Interpretation of DNS

Observations:

- In the finite case, the functional interpretation obtained from this proof of

$$(\neg\neg P_0 \wedge \neg\neg P_1) \to \neg\neg(P_0 \wedge P_1)$$

  and Spector's bar-recursive solution coincide.
- Spector's functional interpretation of DNS can be viewed as infinitary version of this interpretation of finite DNS.

Overview

- Introduction
- Functional Interpretation of DNS
- **Modified Realizability and DNS**

Modified Realizability and DNS

We may write DNS as follows:

$$\forall x((P(x) \to \bot) \to \bot) \to (\forall x P(x) \to \bot) \to \bot.$$

Modified realizability asks for a realizer for $\bot$ in:

$$\begin{array}{lll} G & mr & \forall x((P(x) \to \bot) \to \bot) \\ Y & mr & \forall x P(x) \to \bot. \end{array}$$

Note: There can be no realizer for $\bot$, so we mean here actually a realizer for A, after applying A-translation.

Modified Realizability and DNS

The modified realizability interpretation of DNS uses so-called modified bar-recursion, which is strictly stronger than Spector's variant of bar-recursion.

As modified bar-recursion is not effectively majorizable and depends on a continuity principle for Y, it cannot be used to extract effective bounds from ineffective proofs in analysis.

Modified Realizability and DNS

For finite DNS input is:

$$\begin{array}{lll} G_0 & mr & (P_0 \to \bot) \to \bot, \\ G_1 & mr & (P_1 \to \bot) \to \bot, \\ Y & mr & (P_0 \wedge P_1) \to \bot. \end{array}$$

From the above proof of finite DNS, one obtains the following realizer:

$$G_1(\lambda x_1.G_0(\lambda x_0.Y(x_0, x_1))).$$

However, the extension of this idea to the infinite setting requires a form of bar-recursion even stronger than MBR and not even ineffectively majorizable.

Modified Realizability and DNS

The modified realizability interpretation of full DNS corresponds to a more complicated proof of finite DNS:

$$
\dfrac{\neg(P_0 \wedge P_1)^3 \qquad \dfrac{P_1^2 \qquad \dfrac{\dfrac{\dfrac{\neg(P_0 \wedge P_1)^3 \qquad \dfrac{P_0^1 \quad P_1^2}{P_0 \wedge P_1}\,\wedge I}{\bot}\,\to E}{\neg P_0}\,\to I(1) \qquad \dfrac{\neg\neg P_0 \wedge \neg\neg P_1^4}{\neg\neg P_0}\,\wedge E}{\dfrac{\bot}{P_0}\,\textit{ex falso quodlibet}}\,\to E}{P_0 \wedge P_1}\,\wedge E}{\bot}\,\to E
$$

with further applications of *ex falso quodlibet* and the realizer then depending on a realizer for A-translation of *ex falso quodlibet*: $A \to P^A$.

Future work

Future work/open questions:

- Is there an explanation for the differences between functional interpretation and modified realizability interpretation of DNS?
- Why does - for interpretation the simple proof of finite DNS - the extension to the infinite case lead to such different interpretations of full DNS.
- Is there a modified realizability interpretation of DNS (and a a corresponding proof of finite DNS) using only Spector's bar-recursion?