

Enumeration Reducibility with Polynomial Time Bounds

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Nondeterministic Turing Reducibility

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Definition (Cooper,Sasso,McEvoy)

$f \leq_{NT} g$ if there is a non deterministic Turing machine φ such that $f = \varphi^g$.

Definition

$A \leq_T^N B$ (setwise \leq_{NT}) if $s_A \leq_{NT} c_B$.

Lemma

$A \leq_T^N B$ iff $s_A \leq_T c_B$ iff A c.e. in B .

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Definition (Friedberg and Rogers)

$A \leq_e B$ if there exists a c.e. set W such that for all $s \in \Sigma^*$

$$s \in A \quad \text{iff} \quad \exists t [\langle s, t \rangle \in W \ \& \ D_t \subseteq B] .$$

Equivalently: $A \leq_e B$ if $s_A \leq_{NT} s_B$.

Definition (Ladner et al.)

$A \leq_c^{NP} B$ if there exists an **NP** set V and polynomial $p(n)$ such that such that for all $s \in \Sigma^*$

$$s \in A \quad \text{iff} \quad \exists t [|t| \leq p(|s|) \ \& \ \langle s, t \rangle \in V \ \& \ D_t \subseteq B]$$

Equivalently: $A \leq_c^{NP} B$ if $s_A \leq_{NT}^P s_B$.

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Equivalently: $A \leq_c^{NP} B$ if $s_A \leq_{NT}^P s_B$.

Definition (Selman)

$A \leq_e B$ if $\forall X [B \leq_T^N X \Rightarrow A \leq_T^N X]$.

Definition (Selman)

$A \leq_{pe} B$ if $\forall X [B \leq_T^{NP} X \Rightarrow A \leq_T^{NP} X]$.

Question : Is \leq_{pe} the same as \leq_c^{NP} (over **REC**) ?

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Effective Operator Based Reductions

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A reduction \leq_r is **effective operator based** (uniformly recursive) if there exists an effective enumeration of computable operators $\{\Phi_n \mid n \in \omega\}$ such that, for any $A, B \subseteq \Sigma^*$, $A \leq_r B$ iff $A = \Phi_n(B)$ for some $n \in \omega$.

Remark

Both \leq_e and \leq_c^{NP} are effective operator based.

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$A \leq_T^{\text{NP}} B$ if $\text{SA} \leq_{\text{NT}}^{\text{P}} \text{CB}$.

Definition

$A \leq_{T,q}^{\text{NP}} B$ if there exists $R \leq_T^{\text{P}} B$ such that for all $x \in \Sigma^*$,

$$x \in A \quad \text{iff} \quad \exists w [|w| \leq q(|x|) \ \& \ R(x, w)] .$$

Note

- (Cook, Karp) $A \leq_T^{\text{NP}} B$ iff $A \leq_{T,q}^{\text{NP}} B$ for some $q(n) \in \mathcal{P}$.
- W.l.o.g. (for the sake of simplicity) we can suppose that $A \leq_{T,q}^{\text{NP}} B$ is equivalent to $A \leq_T^{\text{NP}} B$ with polynomial time bound $q(n)$ and that, in this case, $A \leq_T B$ (i.e. deterministically) in $\mathcal{O}(2^{q(n)})$ time.

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Definition (Selman)

$A \leq_{pe}^q B$ if $\forall X [B \leq_{T,q}^{NP} X \Rightarrow A \leq_T^{NP} X]$.

Note

$$\leq_{pe} = \bigcap_{q \in \mathcal{P}} \leq_{pe}^q .$$

Definition (Selman)

$A \leq_{pe'}^q B$ if there exists np-T-machine N and a constant k such that $A = N^B$ and, whenever $x \in A$ there exists an accepting computation y of $N^B(x)$ such that, for all $z \in Q^-(N^B, x, y)$, $q(|z|) \leq k \cdot \log|x|$.
(We call such z **relevant** negative queries.)

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$$\leq_{pe'} = \bigcap_{q \in \mathcal{P}} \leq_{pe'}^q .$$

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Equivalence of \leq_{pe} and $\leq_{pe'}$.

Lemma (Selman)

$$\leq_{pe'}^q \subseteq \leq_{pe}^q.$$

Proof.

Suppose that $A \leq_{pe'}^q B$ and $B \leq_{T,q}^{NP} X$. Then there is an np-T-machine N and constant k such that $A = N^B$ and whenever $x \in A$ then every *relevant* negative query z of $N^B(x)$ satisfies $q(|z|) \leq k \cdot \log|x|$. So each such query can be (deterministically) computed in $\mathcal{O}(2^{q(|z|)}) = \mathcal{O}(|x|^k)$ steps. Accordingly, we can define \hat{N} such that $A = \hat{N}^X$ by appropriate simulation of N in conjunction with the np-T-machine witnessing $B \leq_{T,q}^{NP} X$. Thus in general $\forall X [B \leq_{T,q}^{NP} X \Rightarrow A \leq_{pe}^q X]$. In other words $A \leq_{pe}^q B$. □

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Lemma (Selman)

If $A \not\leq_{pe}^q B$ then there exists $C \leq_T A \oplus B$ such that $B \leq_{T,q}^{NP} C$ whereas $A \not\leq_T^{NP} C$ (so that C witnesses $\exists X [B \leq_{T,q}^{NP} X \ \& \ A \not\leq_T^{NP} X]$), i.e. that $A \not\leq_{pe}^q B$.

Corollary

$$\leq_{pe}^q \subseteq \leq_{pe'}^q.$$

Theorem (Selman)

$$\leq_{pe} \equiv \leq_{pe'}.$$

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If $A \not\leq_{pe} B$ then there exists $C \leq_T A \oplus B$ such that $B \leq_T^{NP} C$ whereas $A \not\leq_T^{NP} C$

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Differences between \leq_c^{NP} and \leq_{pe} .

- \leq_c^{NP} is **effective operator based**. So is \leq_{pe}^q for every polynomial $q(n)$. But what about the relation \leq_{pe} ($= \bigcap_{q \in \mathcal{P}} \leq_{\text{pe}}^q$)?
- \leq_{pe} subsumes and is a weakening of \leq_c^{NP} . As seen above, if $A \leq_{\text{pe}} B$ then for every $q(n) \in \mathcal{P}$ there exists np-T-machine N and constant k such that $A = N^B$ and, for all $x \in A$, all *relevant* negative queries z of $N^B(x)$ satisfy $q(|z|) \leq k \cdot \log|x|$.
(Equivalently, if $A \leq_{\text{pe}} B$ then, for every $m \in \omega^+$, there exists N such that all *relevant* negative queries z of $N^B(x)$ have length $\mathcal{O}([\log|x|]^{1/m})$.)
- \leq_{pe} is a maximal transitive subrelation of $\leq_{\text{T}}^{\text{NP}}$ over **REC**.

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Differences between \leq_c^{NP} and \leq_{pe} .

- There exist $A, B \in \mathbf{DTIME}(2^{2^{2^n}})$ such that $A \leq_{\text{pe}} B$ (via the reduction $x \in A$ iff $\log \log x \in B$) whereas $A \not\leq_c^{\text{NP}} B$.
- \leq_{pe} and \leq_c^{NP} coincide over \mathbf{EXP} . Indeed, suppose that $A \leq_{\text{pe}} B$ and $B \in \mathbf{EXP}$. Then $B \in \mathbf{DTIME}(2^{q(n)})$ for some $q(n) \in \mathcal{P}$. Now, by the equivalence of $\leq_{\text{pe}'}^q$ and \leq_{pe}^q , we know that $A \leq_{\text{pe}} B \Rightarrow A \leq_{\text{pe}'}^q B$ so there exists np-T-machine N and constant k such that $A = N^B$ and, for all $x \in A$, all *relevant* negative queries z of $N^B(x)$ satisfy $q(|z|) \leq k \cdot \log|x|$. Accordingly, these negative queries can be (deterministically) computed in $\mathcal{O}(2^{k \cdot \log|x|}) = \mathcal{O}(|x|^k)$ steps. So just construct an appropriate simulation of N^B to witness $A \leq_c^{\text{NP}} B$.

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$\langle \mathbf{REC}_{pe}, \leq \rangle$

- Is an upper semi-lattice with **NP** as zero degree.
- Is not a lattice.
- Is not distributive.
- Is everywhere branching—i.e. every degree branches.
(Looking ahead technique.)
- Is identical to $\langle \mathbf{EXP}_c^{\mathbf{NP}}, \leq \rangle$ over \mathbf{EXP}_{pe} .

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Further Facts concerning $\langle \mathbf{REC}_{pe}, \leq \rangle$

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By the Corollary established above [◀ Back](#) we know that if $A <_{pe} B$ then there exists $C \leq_T A \oplus B$ such that $A \leq_T^{NP} C$ whereas $B \not\leq_T^{NP} C$. So, using np-T-diagonalisations relative to C we can adapt Ladner's *looking back* technique to prove that $\langle \mathbf{REC}_{pe}, \leq \rangle$ is dense. In fact, by a similar adaptation of Ambos-Spies' Join and Meet Lemmas and results on lattice embeddings in the p-T-degrees we obtain the following.

- If $a < b$, then any countable distributive lattice is embeddable in $[a, b]$ via a lattice embedding preserving 0 and a lattice embedding preserving 1.
- If $a < b$, then any nowhere complemented finite distributive lattice is embeddable in $[a, b]$ via a lattice embedding preserving both 0 or 1.

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enumeration
reducible to
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The End