Random Closed Sets

Paul Brodhead,
Douglas Cenzer,
Seyyed Dashti
University of Florida

CIE 2006, Swansea

Kolmogorov/Chaitin Randomness

- A random real is *incompressible*.
- A partial function $F : \{0,1\}^* \to \mathbb{N}$ is *prefix-free* if whenever $F(\sigma)\downarrow$ and $\sigma \sqsubset \tau$, then $F(\tau)\uparrow$.
- There is a universal computable prefix-free U.
- For $\tau \in \{0,1\}^*$,

  $K(\tau) = min\{|\sigma| : U(\sigma) = \tau\}$.
- $X \in \{0,1\}^{\mathbb{N}}$ is *(Chaitin) random* if there is a constant c such that $K(X\lceil n) \geq n - c$ for all n.
- Chaitin's $\Omega = \sum_{\sigma \in Dom(U)} 2^{-|\sigma|}$.

  This is a random real.
- There is a random real with $K(X\lceil n) \geq n$ for all n.

  This is easily modified for $X \in \{0,1,\ldots,k\}^{\mathbb{N}}$.

Martin-Lof Randomness

- A random real does not belong to a Π^0_2 class with effective measure 0.

- Martin-Lof Test: A computable sequence U_n of effectively open sets with $\mu(U_n) \leq 2^{-n}$ for all n.

  (More generally, $\mu(U_n) \leq t_n$ for some computable sequence with limit 0.)

  X is Martin-Lof Random if for any Martin-Lof Test $\{U_n\}_{n<\omega}$, $X \notin \cap_n U_n$.

- Martin-Lof: The set of Martin-Lof random reals has measure one.

- Schnorr: X is Martin-Lof random if and only if X is Chaitin random.

C.E. Reals and Randomness

- Let Q be the set of rationals.

  A real r is *c.e.* if $\{q \in Q : q < r\}$ is c.e.

- Theorem: The following are equivalent.

  (a) r is c.e.;

  (b) r is the limit of a computable increasing sequence of rationals;

  (c) $r = \sum_{\sigma \in S} 2^{-|\sigma|}$ for some computable prefix-free set S;

  (d) $r = \sum_{\sigma \in S} 2^{-|\sigma|}$ for some c.e. prefix-free set S;

  (e) $r = m(V)$ for some c.e. open set V.

- Thus Chaitin's Ω is a random c.e. real.

Closed Sets and Trees

- $\{0,1\}^*$ is the set of finite strings on $\{0,1\}$.

- $\sigma \sqsubseteq \tau$ for $\sigma = (\sigma(0), \ldots, \sigma(m))$ and $\tau = (\tau(0), \ldots, \tau(n))$ if $m \leq n$ and $\sigma(i) = \tau(i)$ for $i \leq m$.

- $T \subseteq \{0,1\}^*$ is a *tree* if $\tau \in T$ and $\sigma \sqsubseteq \tau$ implies $\sigma \in T$.

- For $X \in \{0,1\}^{\mathbb{N}}$, $n \in \mathbb{N}$, $X \lceil n = (X(0), \ldots, X(n-1))$.

  $\sigma \sqsubset X$ iff $\sigma = X \lceil n$ for some n.

  Topology on $\{0,1\}^{\mathbb{N}}$ is generated by *intervals* $I(\sigma) = \{X : \sigma \sqsubset X\}$.

- $X \in [T] \iff (\forall n) X \lceil n \in T$.

- FACT: Q is closed iff $Q = [T_Q]$ for some tree T.

  $T_Q = \{\sigma : I(\sigma) \cap Q \neq \emptyset\}$.

Measure on Space of Closed Sets

- $\mathcal{C}$ is the space of closed subsets Q of $\{0,1\}^{\mathbb{N}}$.
- Topology on $\mathcal{C}$ is generated by sets $K(\sigma) = \{K : K \cap I(\sigma) \neq \emptyset\}$.
- Define measure μ^* recursively on $K(\sigma)$.
- $\mu^*(K(\emptyset)) = 1$. (Q is nonempty with probability 1.)
- $\mu^*(K(\sigma^\frown 0)) = \mu^*(K(\sigma^\frown 1)) = \frac{2}{3}\mu^*(K(\sigma))$ and $\mu^*(K(\sigma^\frown 0) \cap K(\sigma^\frown 1)) = \frac{1}{3}\mu^*(K(\sigma))$.
- For $\sigma \in T_Q$, $\sigma^\frown i \in T_Q$ with probability $\frac{2}{3}$ for each i and *both* are in T_Q with probability $\frac{1}{3}$.
- These fractions are arbitrary–one could use $\frac{3}{4}$ and $\frac{1}{2}$, for example.

Representation of Closed Sets

- A nonempty closed set $Q = [T_Q]$ is coded by the real $X = X_Q$ as follows.

  Let $\sigma_0 = \emptyset, \sigma_1, \ldots$ enumerate T_Q in order, first by length and then lexicographically.

  Let $X(n) = 2$ if $\sigma_n{}^\frown 0 \in T_Q$ and $\sigma_n{}^\frown 1 \in T_Q$;

  Let $X(n) = i \in \{0, 1\}$ if $\sigma_n{}^\frown i \in T_Q$ and $\sigma_n{}^\frown(1-i) \notin T_Q$.

- For $\mathcal{A} \subseteq \mathcal{C}$, $\mu^(\mathcal{A}) = \mu(\{X_Q \in \{0, 1, 2\}^* : Q \in \mathcal{A}\}$.

- This representation is a one-to-one correspondence between $\{0, 1, 2\}^{\mathbb{N}}$ and $\mathcal{C}$.

- $Q \in \mathcal{C}$ is a *random closed set* if X_Q is a random real.

- Theorem 1. There exists a random Π^0_2 closed set, corresponding to a c.e. random representation X.

Members of Random Closed Sets

- Lemma 1. If Q is a random closed set, then $Q \cap I(\sigma)$ is random for any $\sigma \in T_Q$.

- Theorem 2. If Q is a random closed set, then Q has no computable members.

- Sketch: Let Q be a random closed set and fix a computable real X.

  Let $\mathcal{A}_n = \{P : P \cap I(X\lceil n) \neq \emptyset\}$.

  Each $\mathcal{A}_n$ is a clopen set in $\mathcal{C}$.

  Then $\mu^*(\mathcal{A}_n) = (\frac{2}{3})^n$ for each n, so this is a Martin-Lof Test.

  Hence $Q \notin \mathcal{A}_n$ for some n, so that $X \notin Q$.

Random Closed Sets are Perfect

- Theorem 3. If Q is a random closed set, then Q has no isolated elements.

- Sketch: By Lemma 1, it suffices to show that Q is not a singleton.

  For each n, let

  $\mathcal{A}_n = \{P : card(\{\sigma \in \{0, 1\}^n : P \cap I(\sigma) \neq \emptyset\}) = 1\}$.

  That is, $P \in \mathcal{A}_n$ if T_P has one node of length n.

  P is a singleton if and only if $P \in \cap_n \mathcal{A}_n$.

  Again $\mathcal{A}_n$ is clopen and $\mu^*(\mathcal{A}_n) = (\frac{2}{3})^n$.

  Thus $Q \notin \mathcal{A}_n$ for some n.

  It follows that a random closed set must be perfect.

Random Sets Have Measure Zero

- Theorem 4. If Q is a random closed set, then $\mu(Q) = 0$.

- Sketch: For each m, show that $\{P : \mu(P) \geq 2^{-m}\}$ has measure 0. This is by induction on m.

  For a tree T, let $T^\ell = T \cap \{0,1\}^\ell$. Then $\mu([T]) = \lim_\ell \mu([T^\ell])$.

  For each fixed m and $P = [T_P]$, $\mu(P) \geq 2^{-m}$ iff $\mu([T^\ell]) \geq 2^{-m}$ for all ℓ.

  This Martin-Lof Test shows that $\mu(Q) < 2^{-m}$.

Π^0_1 Classes Are Not Random

- Lemma 2. For any $Q \in \mathcal{C}$, $\mu^*(\{P : P \subseteq Q\}) \leq \mu(Q)$.

- Theorem 5. Let Q be a Π^0_1 class with measure 0. Then no subset of Q is random.

- Sketch: Let $Q = [T]$ where T is computable (possibly with dead ends) and assume $\mu(Q) = 0$. Then $\mu([T^\ell])$ is a computable sequence with limit 0.

  Thus $\mathcal{A}_\ell = \{P : P \subseteq [T^\ell]$ is a Martin-Lof Test.

  Hence if P is random, then P is not a subset of some $[T^\ell]$ and thus is not a subset of Q.

  But every random class has measure 0. Thus

- Theorem 6. No Π^0_1 class is random.

Conclusions and Future Research

- In this paper, we define the notion of a random closed set and derive a few properties.

- We can also define random continuous functions.

- Compressibility of trees is also interesting. The natural coding of T^ℓ has length 2^ℓ as a characteristic function. But in fact for any tree T, there are constants c and k such that

  $K(T^\ell) \leq 2^\ell - 2^{\ell-k} + c$.

- There is a Π^0_1 $P = [T]$ with $K(T^\ell) \geq \ell$ for all ℓ.

  We will continue to examine the connection between $K(T^\ell$ and the randomness of $[T]$.