

# Reducibility Between Domain Representations

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- 1 Introduction
  - Domain Representations — Why? and What?
  - Choosing Representations
- 2 Reductions Between Domain Representations
- 3 Results

# Domain Representations — Why?

- Computations on uncountable data requires computations on **approximations**.
- In general, spaces do not contain **proper** approximations.
- Domains can be constructed containing **both** the space and finite approximations.
- The natural **effectivity** theory of the domain can be exported to the space.

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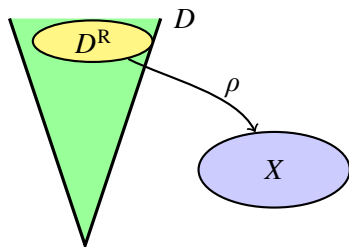
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# Domain Representations — What?

## Definition

A **representation** of a topological space  $X$  is a triple  $(D, D^R, \rho)$ , where  $D$  is a domain,  $D^R \subseteq D$ , and where  $\rho : D^R \rightarrow X$  is onto and continuous.

The elements of  $D^R$  contain **total** information about an element of  $X$  and  $D^R$  is therefore often referred to as a **totality** on  $D$ .



# Representations are Important

- Recursive versions of Cauchy sequences, Dedekind cuts, decimal expansion, etc., all give the **same class of computable reals**.
- However, the class of **computable operations may differ**.
- For example, neither addition nor multiplication is computable on the space of reals with computable decimal expansions.
- The computable version of Cauchy sequences make all common continuous operations computable.

## Question

*Is there a framework where the structural properties of representations can be understood, so that algebras over other uncountable spaces can be given a good computability theory?*

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# Choosing a Domain Representation

When choosing between different domain representations of a space we can:

- look at their intrinsic properties, or
- look at how they relate to other possible representations.

**Reductions** will be used for the latter approach.

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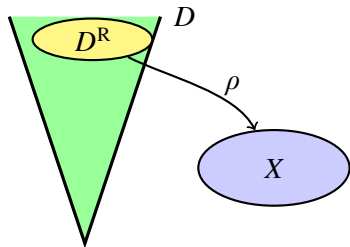
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# Intrinsic Properties of Domain Representations

- The type of the domain  $D$ .
- Observing how  $\rho$  relates the topologies of  $D^{\mathbb{R}}$  and  $X$ .
  - $\rho$  continuous
  - $\rho$  a quotient mapping
  - $\rho$  retraction
  - ...
- Properties of the totality  $D^{\mathbb{R}}$ .
  - $D^{\mathbb{R}}$  **dense** in  $D$
  - $D^{\mathbb{R}}$  **upwards-closed**, i.e.  $x \in D^{\mathbb{R}}$  and  $x \sqsubseteq y$  implies  $y \in D^{\mathbb{R}}$  and  $\rho(y) = \rho(x)$ .

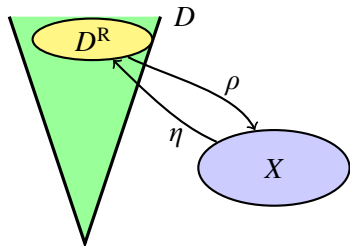


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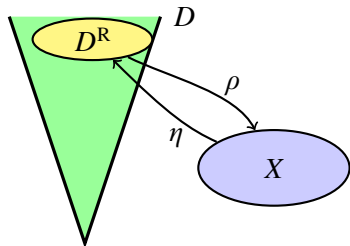


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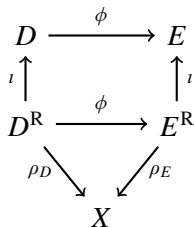
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# Reductions Between Domain Representations

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  - Definition of Reductions
  - Spectrums of Domain Representations
- 3 Results

# Definition of Reductions



- $D \leq_c E$
- $\phi$  represents  $\text{id}_X$
- $\phi$  translates  $D$  to  $E$

## Definition

Let  $(D, D^R, \rho_D)$  and  $(E, E^R, \rho_E)$  be representations of a topological space  $X$ , and let  $\phi : D \rightarrow E$  satisfy  $\phi[D^R] \subseteq E^R$  and  $\rho_D(d) = \rho_E\phi(d)$ .

- 1  $\phi$  is a **continuous reduction** of  $D$  to  $E$ , written  $D \leq_c E$ , if  $\phi$  is continuous.
- 2  $\phi$  is a **continuous partial reduction** of  $D$  to  $E$ , written  $D \leq_{cp} E$ , if  $\phi$  is a partial continuous function such that  $D^R \subseteq \text{dom } \phi$ .

# Spectrums of Domain Representations

- $\leq_c$  and  $\leq_{cp}$  are pre-orders on  $\mathcal{D}$ .
- Let  $\mathcal{D}$  be a class of domain representations of a space  $X$ , e.g., the class  $\mathbf{DRep}(X)$  of all domain representations of  $X$ .

## Definition

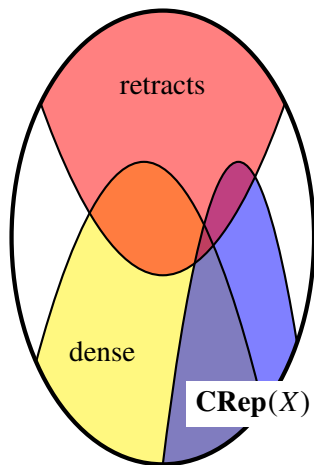
A **spectrum** over a topological space  $X$ , written  $\text{Spec}(X, \mathcal{D}, \leq)$ , is the quotient  $\mathcal{D}/\equiv$  ordered by  $\leq$ , where  $\leq$  is some pre-order on  $\mathcal{D}$ .

## Definition

A domain representation  $D$  is **universal** in  $\text{Spec}(X, \mathcal{D}, \leq)$  if every  $E \in \mathcal{D}$  reduces to  $D$ .

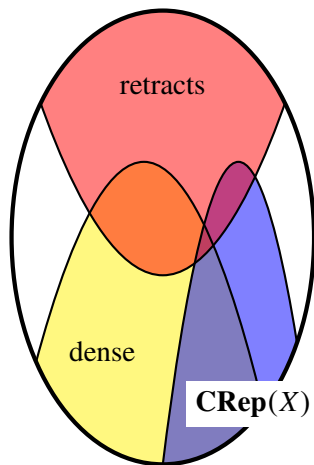
# The spectrum $\text{Spec}(X, \mathbf{DRep}(X), \leq_c)$

- Dense representations reduce to retracts.
- Dense retracts are unique up to  $\equiv_c$ .
- Spectrum usually non-trivial.
- $\mathbf{CRep}(X)$  of Cantor domain representations is a subclass of  $\mathbf{DRep}(X)$ .
- Using  $\leq_{cp}$  would partially collapse diagram since  $D \equiv_{cp} D^D$ .



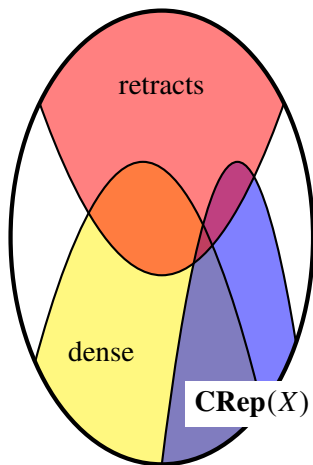
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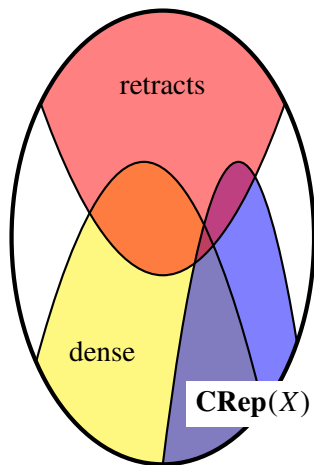
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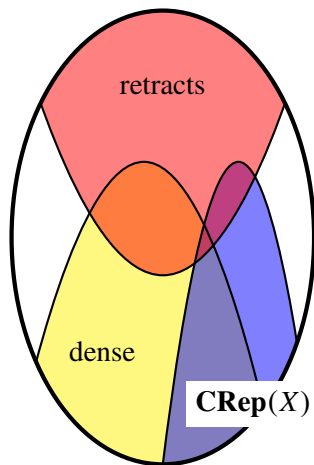
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  - Cantor–Weihrauch Domain Representations
  - Admissibility

# Cantor–Weihrauch Domain Representations

## Definition

A CW-representation of a space  $X$  is a domain representation  $(\mathcal{C}, \text{dom } \delta, \delta)$ , where  $\mathcal{C}$  is the Cantor domain, and  $\delta$  is Weihrauch's standard representation of a countably based  $T_0$  space.

It is known that CW-representations are universal in  $\text{Spec}(X, \mathbf{CRep}(X), \leq_c)$ .

## Theorem

*CW-representations are continuously equivalent to dense retract domain representations.*

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# Representations of Real Numbers

## Theorem

*The representations  $\mathcal{R}$  (the interval domain) and  $\mathcal{R}_{\text{cda}}$  (subdomain of centred dyadic approximations) are equivalent.*

## Note

$\mathcal{R}_{\text{cda}}$  is a bifinite domain that fails to be consistently complete. It is nevertheless admissible since  $\mathcal{R}$  is admissible.

# Summary

- Reductions relate representations to each other.
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