

Complexity of Graph Isomorphism for Restricted Graph Classes

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Interdisciplinary Character of Graph Isomorphism

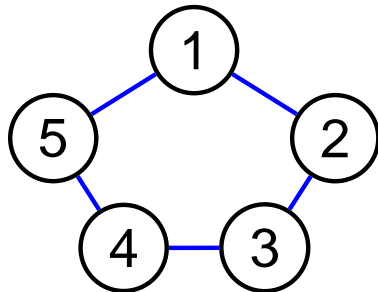
- Combinatorial techniques
- Algebraic techniques, group theory
- Descriptive complexity, logic
- Counting classes
- Lowness
- Arthur-Merlin games
- Derandomization techniques
- Interactive proofs, zero knowledge
- ⋮

Graphs and Hypergraphs

Let $V = \{1, \dots, n\}$.

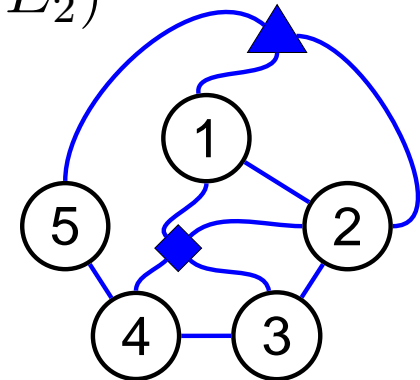
- $G = (V, E)$ is a **graph**, if $E \subseteq \binom{V}{2}$.
- $G = (V, E)$ is a **hypergraph**, if $E \subseteq \mathcal{P}(V)$.

$$G_1 = (V, E_1), V = \{1, \dots, 5\}$$



$$E_1 : \{1, 2\}, \{2, 3\}, \{3, 4\}, \\ \{4, 5\}, \{1, 5\}$$

$$G_2 = (V, E_2)$$



$$E_2 : \{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \\ \{1, 2, 5\}, \{1, 2, 3, 4\}.$$

Isomorphisms and Automorphisms

For a permutation f on V and a hyperedge $e = \{v_1, \dots, v_k\} \subseteq V$ let

$$f(e) = \{f(v_1), \dots, f(v_k)\}.$$

Let $G = (V, E)$ and $H = (V, E')$ be hypergraphs.

- f is an **isomorphism** between G and H , if

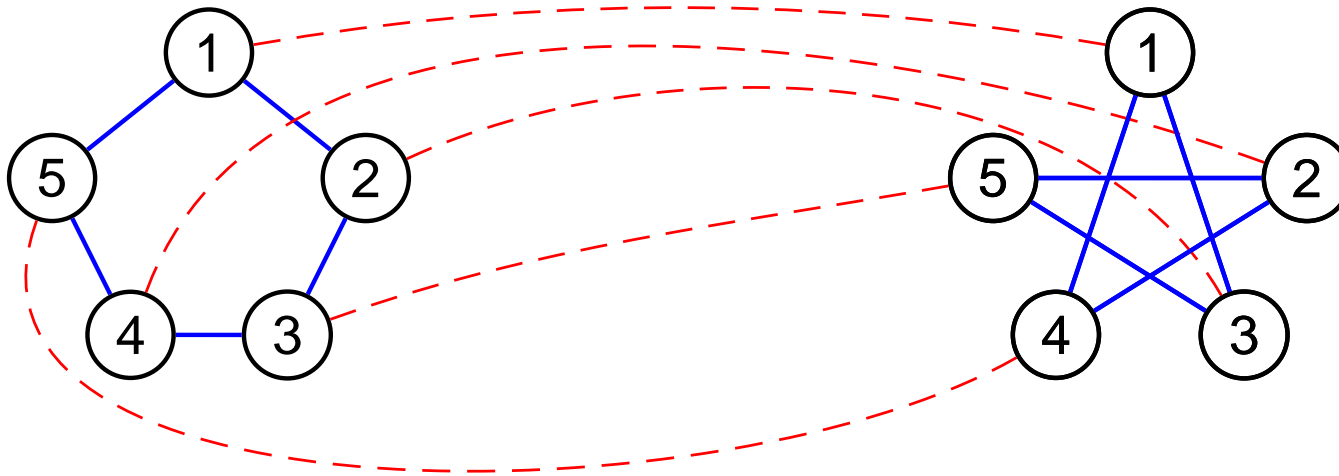
$$\forall e \subseteq V : e \in E \Leftrightarrow f(e) \in E'.$$

- f is an **automorphism** of G , if

$$\forall e \in E : f(e) \in E.$$

An Isomorphism Between Graphs

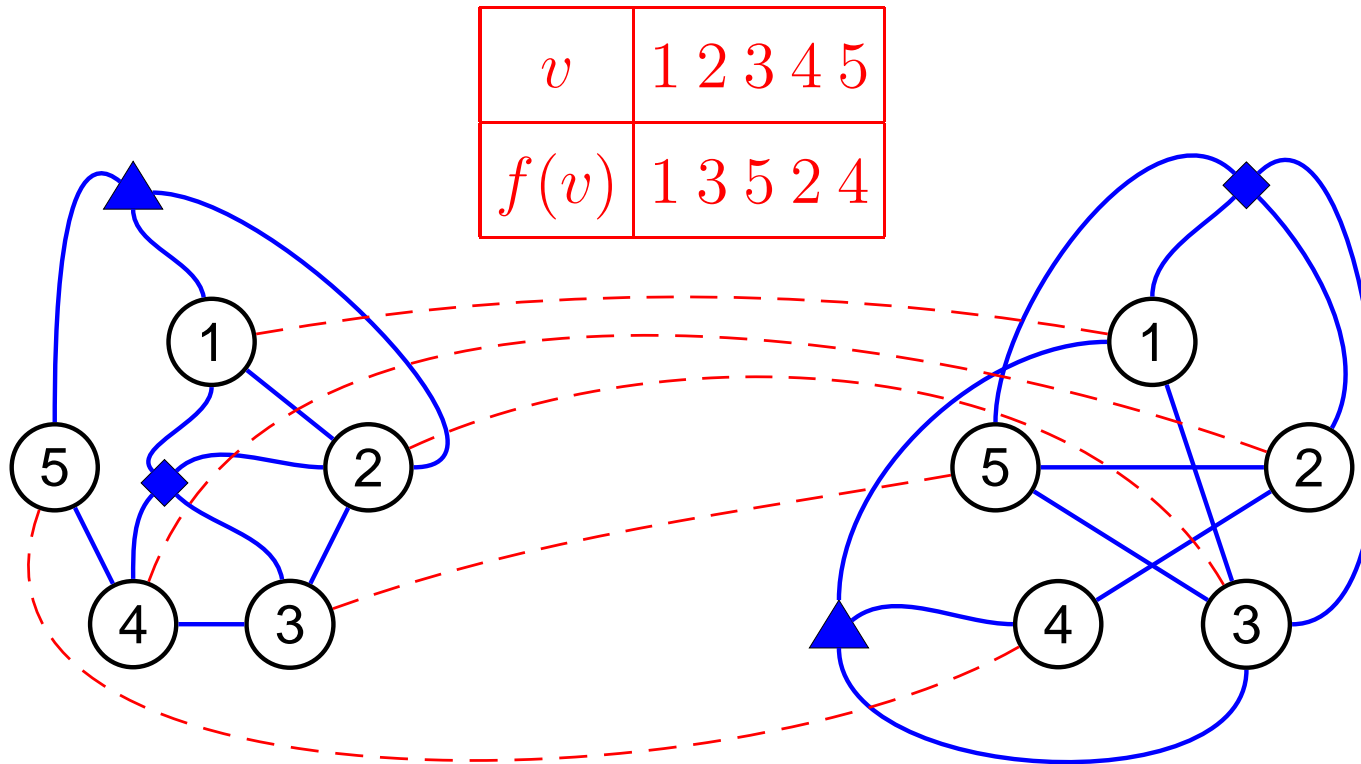
v	1 2 3 4 5
$f(v)$	1 3 5 2 4



$$G = (V, E)$$

$$f(G) = (V, \underbrace{\{f(e) \mid e \in E\}}_{f(E)})$$

An Isomorphism Between Hypergraphs



$E: \{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{1, 2, 5\}, \{1, 2, 3, 4\}$

$f(E): \{1, 3\}, \{3, 5\}, \{2, 5\}, \{2, 4\}, \{1, 3, 4\}, \{1, 2, 3, 5\}$

Complexity of GI for Restricted Graph Classes

Deciding graph isomorphism (GI) is in P for restricted graph classes as, e.g.

- **trees** and **planar** graphs [Hopcroft, Tarjan 71]
- graphs of **bounded genus** [Miller 80]
- graphs of **bounded degree** [Luks 82]
- graphs of **bounded eigenvalue multiplicity** [Babai, Grigoryev, Mount 82]

However, the restriction to

- **regular** or **bipartite** graphs

does not decrease the complexity of GI.

Complexity Classes

$AM = BP \cdot NP$,

NC^i : uniform $\{\wedge, \vee, \neg\}$ -circuits of polynomial size and depth $O(\log^i n)$,

TC^i : like NC^i but additionally with threshold gates,

AC^i : like NC^i but with unbounded fanin gates,

$NC = \bigcup_{i \geq 0} NC^i$,

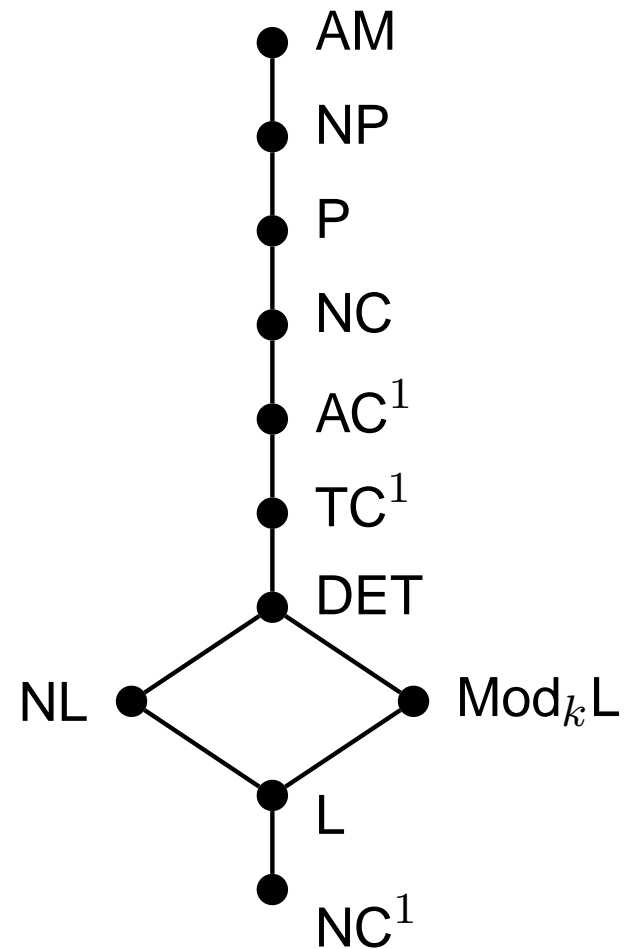
$DET = NC^1(\text{INT-DETERMINANT})$,

$L = DSPACE(\log n)$, ($L = SL$ [Reingold 04])

$NL = NSPACE(\log n)$,

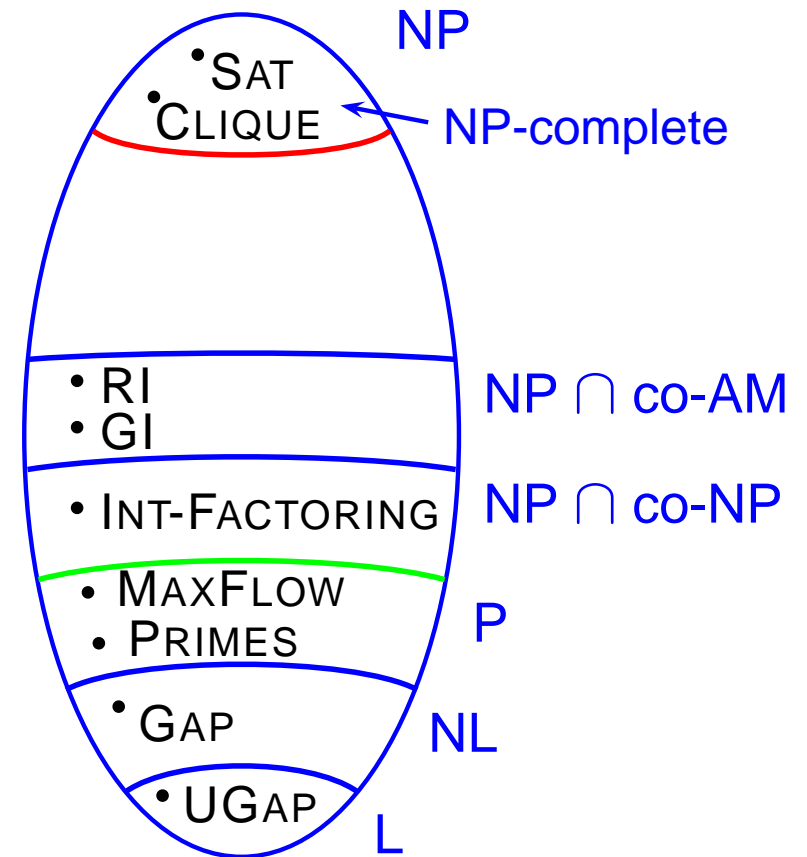
$\text{Mod}_k L$ contains all sets A s.t. there is a nondet. logspace machine M with

$$A = \{x \mid \#\text{accepting paths of } M(x) \equiv 1 \pmod{k}\}$$



Complexity of GI

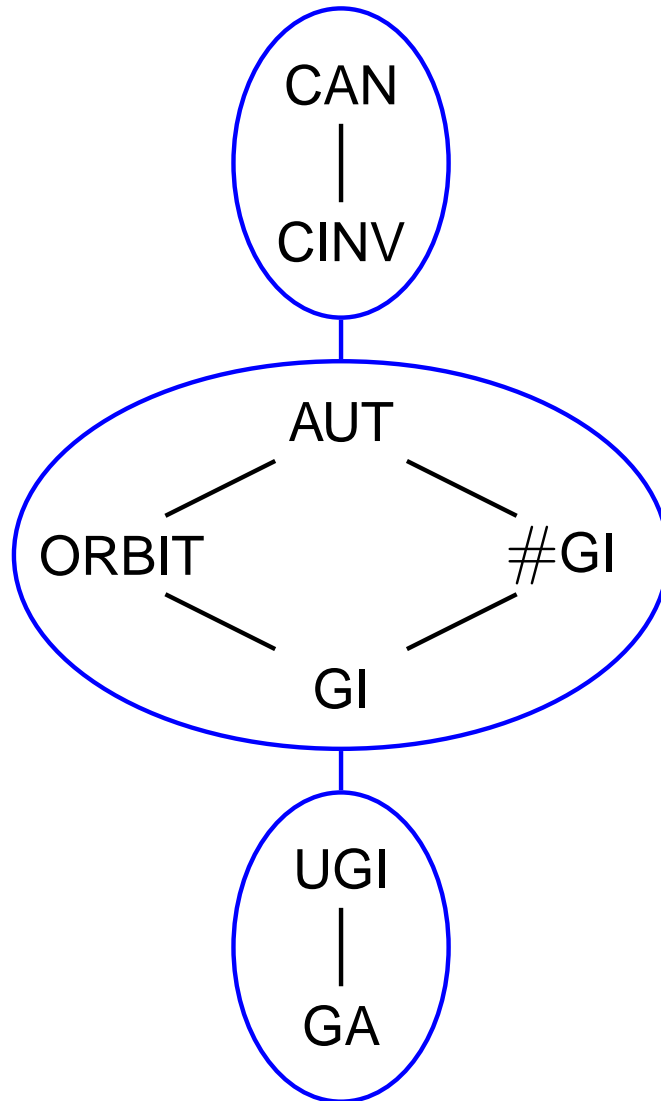
- $GI \in \text{co-AM}$ [Goldwasser, Sipser 87]
- $GI \in \text{SPP}$ [Arvind, Kurur 02]
- GI is hard for DET [Torán 00]
- $GI \in \text{NPC} \Rightarrow \text{PH} = \Sigma_2^p$
[Boppana, Hastad, Zachos 87]
- $GI \in \text{DTIME}(exp(O(\sqrt{n \log n})))$
[Luks, Zemlyachenko 83]



Problems Related to GI

- GA:** Does G have a non-trivial automorphism?
- UGI:** Is there a unique isomorphism between G and H ?
- #GI:** Determine the number of isomorphisms between G and H .
- AUT:** Compute a generating set for the automorphism group of G .
- ORB:** Determine the orbits of the automorphism group of G .
- CINV:** Compute a complete invariant f for G , i.e.,
for all graphs G and H , $G \cong H \Leftrightarrow f(G) = f(H)$.
- CAN:** Compute a canonization g for G , i.e.,
for all graphs G and H , $G \cong g(G) \wedge [G \cong H \Rightarrow g(G) = g(H)]$.

Relative Complexity of Problems Related to GI



$$\text{CAN} \leq_T^P \text{CINV} \quad [\text{Gurevich 97}]$$

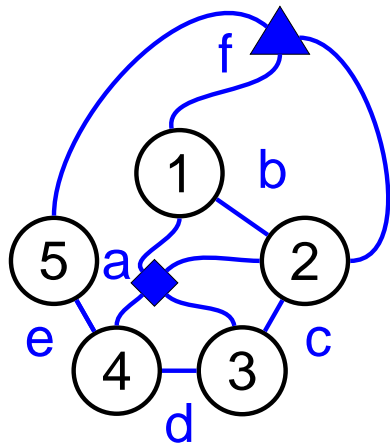
$$\text{AUT} \leq_T^P \text{GI} \quad [\text{Mathon 79}]$$

Relative Complexity of GI and HGI

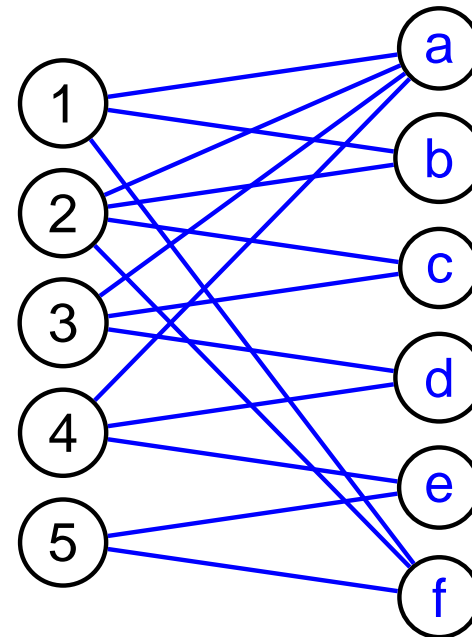
HGI: Are hypergraphs G and H isomorphic?

HGA: Does hypergraph G have a non-trivial automorphism?

$$G = (V, E)$$



$$G' = (V \cup E, \{\{v, e\} \mid v \in e\})$$



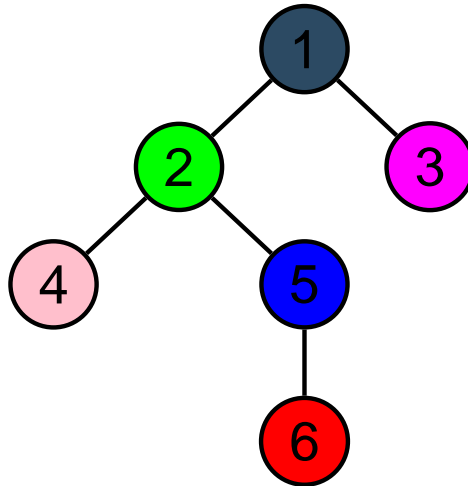
Then we have:

$$GI \equiv HGI \text{ and } GA \equiv HGA.$$

Representing Trees

We consider two different representations of trees:

- by a **string** of nested parentheses,
- by a **pointer list**, i.e., a sequence of edges (ordered pairs).



string representation:

$(((() (())) ()))$

pointer list:

$(2,5), (2,4), (1,2), (1,3), (5,6)$

Complexity of Tree Isomorphism

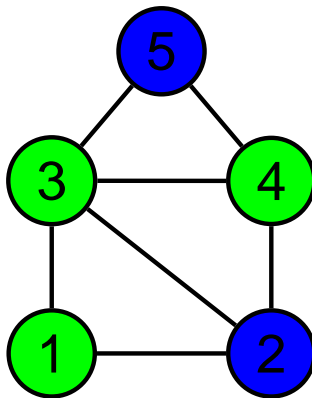
By restricting GI to trees in the string or pointer list representation we obtain the problems **STRING-TI** and **POINTER-TI**.

- **POINTER-TI** \in LINTIME [Aho, Hopcroft, Ullman 74]
- **POINTER-TI** \in NC [Miller, Reif 91]
- **POINTER-TI** \in L [Lindell 92] (even canonization)
- **STRING-TI** \in NC¹ [Buss 97] (even canonization)
- **BOUNDED-TREEWIDTH-GI** \in TC¹ [Grohe, Verbitsky 06] (even CINV)
- **POINTER-TI** and **STRING-TI** are also hard for L and NC¹, respectively [Jenner, K, McKenzie, Torán 03]

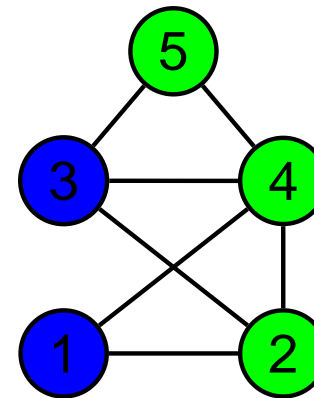
GI for Colored Graphs

COLOR-GI is the problem GI applied to graphs with node colors $c(v)$, which have to be preserved by the isomorphism f ,

$$c(f(v)) = c(v).$$



v	1 2 3 4 5
$f(v)$	5 3 4 2 1



Complexity of GI for Bounded Color Classes

By restricting COLOR-GI to graphs having at most k nodes of the same color we obtain the problem $\text{COLOR}_k\text{-GI}$.

- $\text{COLOR}_{O(1)}\text{-GI} \in \text{RP}$ [Babai 79]
- $\text{COLOR}_{O(1)}\text{-GI} \in \text{P}$ [Furst,Hopcroft,Luks 80]
- $\text{COLOR}_{O(1)}\text{-GI} \in \text{NC}^4$ [Luks 86]
- $\text{COLOR}_{O(1)}\text{-GI} \in \text{NC}^2$ [Arvind, Kurur, Vijayaraghavan 05]

Complexity of Color_k-GI for Small k



⊕L-complete
[Arvind, K 06]



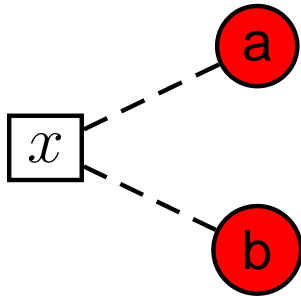
L-complete
[Jenner, K, McKenzie, Torán 03]

Known Upper and Lower Bounds

Problem	hard for	inside
STRING-TA, STRING-CAN	NC^1	NC^1
POINTER-TA, POINTER-CAN	L	L
COLOR ₂ -GA, COLOR ₃ -CAN	L	L
COLOR ₄ -GA, COLOR ₅ -GI	$\oplus L$	$\oplus L$
COLOR ₂ -HGA, COLOR ₂ -HGI	$\oplus L$	$\oplus L$
COLOR _p -HGA, COLOR _p -HGI, p prime	$\text{Mod}_{p!}L$	P
GA, GI	DET	$NP \cap \text{co-AM}$

Warm-Up: COLOR₂-GA ∈ L

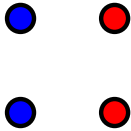
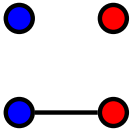
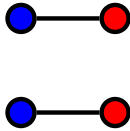
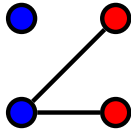
The 1-Bit Representation for Color Classes of Size 2



a	b	x
a	b	0
b	a	1

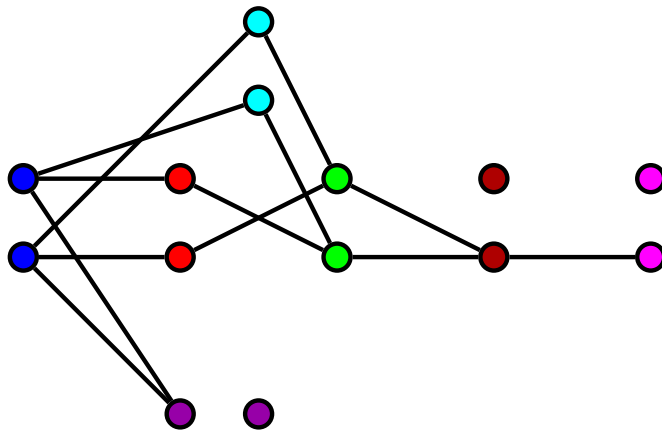
Warm-Up: Color₂-GA ∈ L

Transform graph G with color classes of size 2 into a system S of equations of the form $x_i = x_j$ and $x_i = 0$:

Edges in G between C_i and C_j			
			
-	$x_i = x_j = 0$	$x_i = x_j$	$x_i = 0$
Equations in S			

Then G has a non-trivial automorphism iff S has a non-zero solution.

Warm-Up: Color₂-GA ∈ L



G

$$x_1 = x_2$$

$$x_2 = x_3$$

$$x_3 = x_4$$

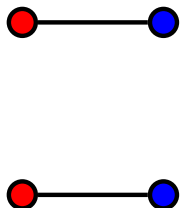
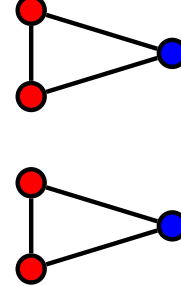
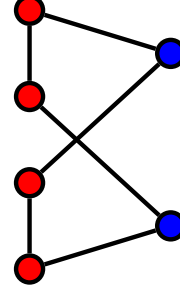
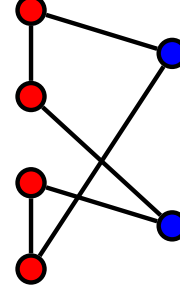
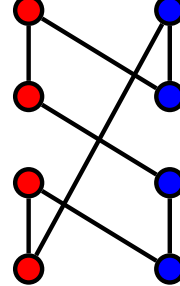
$$x_5 = x_6 = 0$$

$$x_7 = 0$$

S

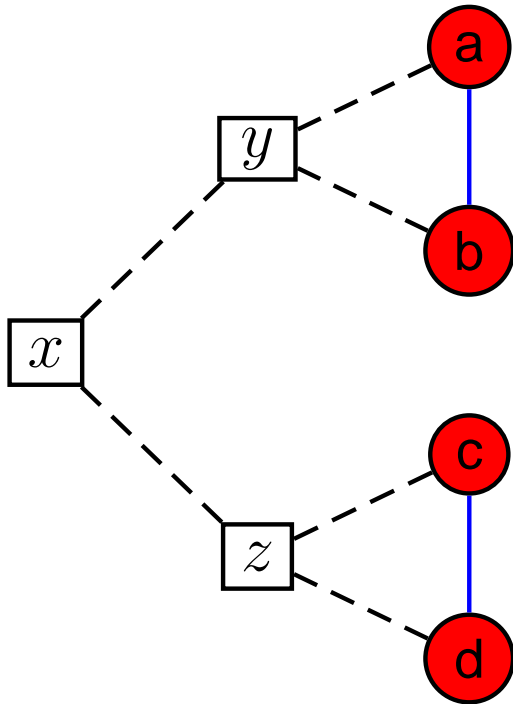
COLOR₄-GA $\in \oplus L$

Also COLOR₄-GA can be reduced to solving a system S of equations (this time over \mathbb{F}_2):

Edges in G between C_i and C_j and the corresponding equations				
				
$x_i = x_j$	$x_i = x_j$	$y_i = z_i = x_j$	$x_i = x_j \oplus y_i,$ $y_i = z_i$	$x_i = x_j \oplus y_j,$ $y_i = z_i = y_j = z_j$

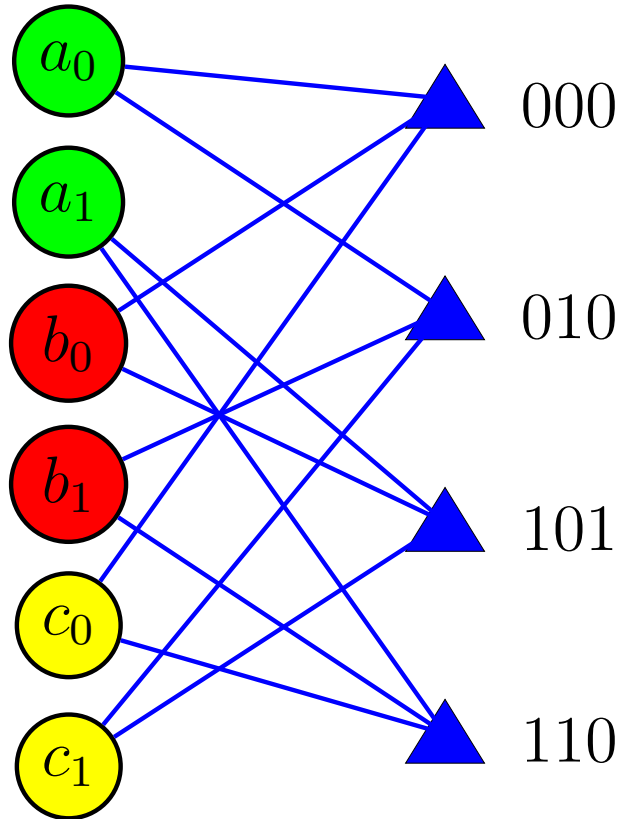
The 3-Bit Representation for Halved Color Classes

A color class of size 4 is called **halved**, if it contains two disjoint edges.



<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	<i>x</i>	<i>y</i>	<i>z</i>
<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	0	0	0
<i>a</i>	<i>b</i>	<i>d</i>	<i>c</i>	0	0	1
<i>b</i>	<i>a</i>	<i>c</i>	<i>d</i>	0	1	0
<i>b</i>	<i>a</i>	<i>d</i>	<i>c</i>	0	1	1
<i>c</i>	<i>d</i>	<i>a</i>	<i>b</i>	1	0	0
<i>d</i>	<i>c</i>	<i>a</i>	<i>b</i>	1	0	1
<i>c</i>	<i>d</i>	<i>b</i>	<i>a</i>	1	1	0
<i>d</i>	<i>c</i>	<i>b</i>	<i>a</i>	1	1	1

COLOR₂-HGA $\in \oplus L$



$$x \in \text{Aut}(G) \Leftrightarrow \forall e \in E : x \oplus e \in E$$

Open Problems

- Is GI for graphs of bounded treewidth decidable in L?
- Do these graphs admit an NC (or even TC^1) canonization?
- Can we close the gap between the upper and lower complexity bounds for $COLOR_k$ -GI, $k \geq 6$, and for $COLOR_k$ -HGI, $k \geq 3$?
- In particular, are $COLOR_k$ -HGA and $COLOR_k$ -HGI in NC?
- Can Gurevich's reduction of canonization to computing a complete invariant be implemented in NC? (At least for special graph classes?)

THANK YOU