Complexity of Graph Isomorphism for Restricted Graph Classes

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Interdisciplinary Character of Graph Isomorphism

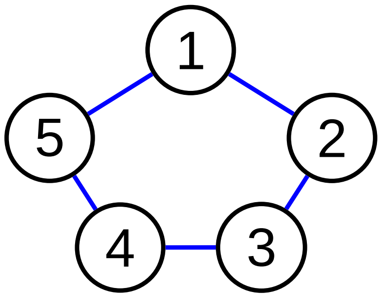
- Combinatorial techniques
- Algebraic techniques, group theory
- Descriptive complexity, logic
- Counting classes
- Lowness
- Arthur-Merlin games
- Derandomization techniques
- Interactive proofs, zero knowledge
- ⋮

Graphs and Hypergraphs

Let $V = \{1, \ldots, n\}$.

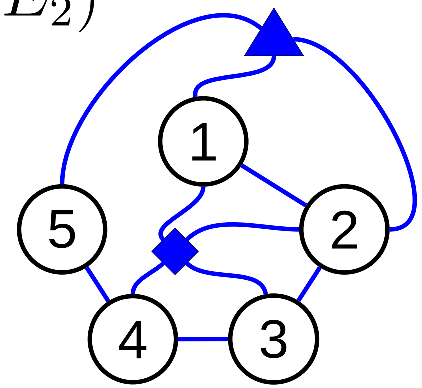
- $G = (V, E)$ is a **graph**, if $E \subseteq \binom{V}{2}$.
- $G = (V, E)$ is a **hypergraph**, if $E \subseteq \mathcal{P}(V)$.

$G_1 = (V, E_1), V = \{1, \ldots, 5\}$



$E_1: \quad \{1,2\}, \{2,3\}, \{3,4\},$
$\{4,5\}, \{1,5\}$

$G_2 = (V, E_2)$



$E_2: \quad \{1,2\}, \{2,3\}, \{3,4\}, \{4,5\},$
$\{1,2,5\}, \{1,2,3,4\}.$

Isomorphisms and Automorphisms

For a permutation f on V and a hyperedge $e = \{v_1, \ldots, v_k\} \subseteq V$ let

$$f(e) = \{f(v_1), \ldots, f(v_k)\}.$$

Let $G = (V, E)$ and $H = (V, E')$ be hypergraphs.

- f is an **isomorphism** between G and H, if

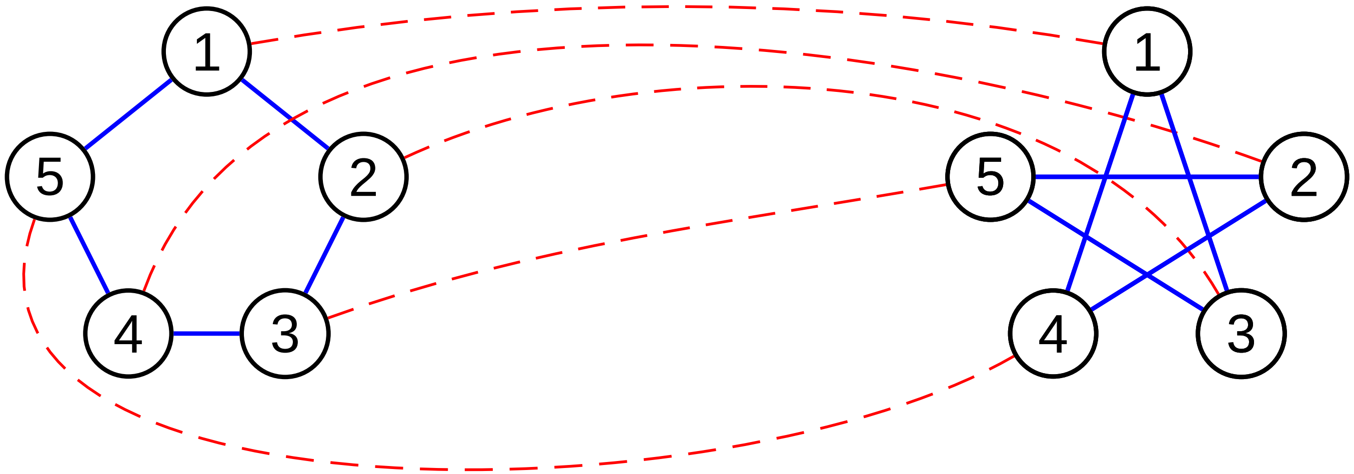
$$\forall e \subseteq V : e \in E \Leftrightarrow f(e) \in E'.$$

- f is an **automorphism** of G, if

$$\forall e \in E : f(e) \in E.$$

An Isomorphism Between Graphs

v	$1\;2\;3\;4\;5$
$f(v)$	$1\;3\;5\;2\;4$

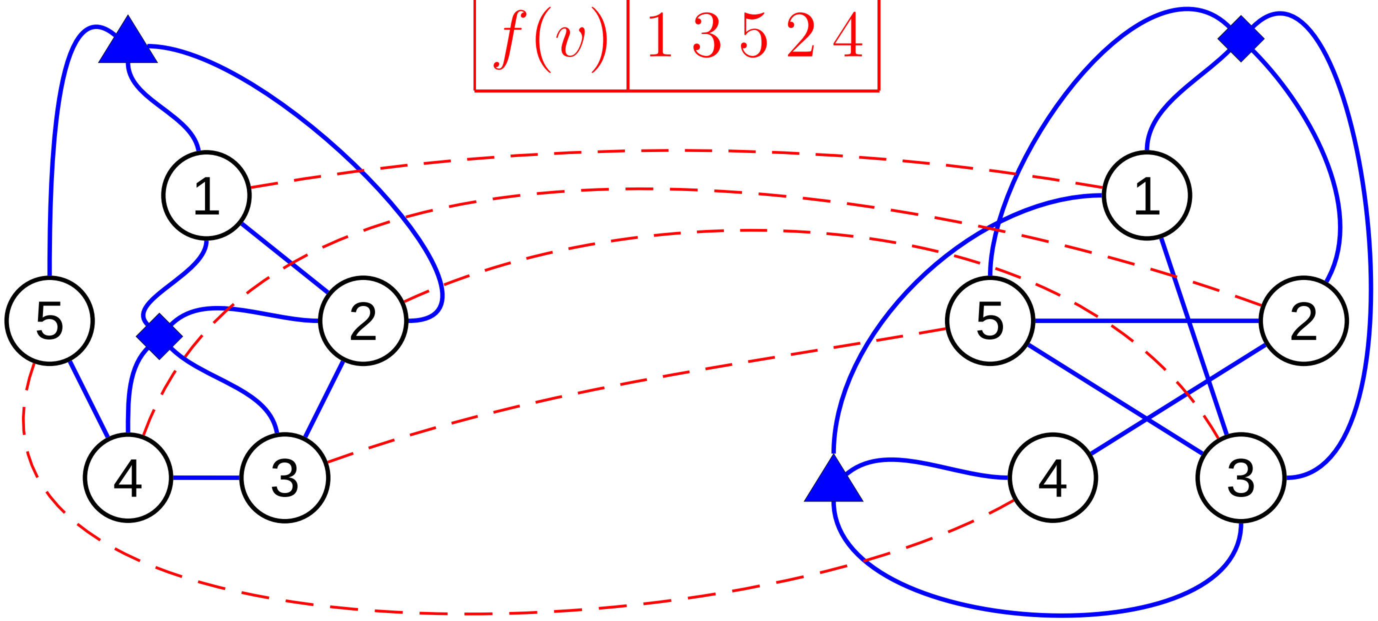


$G = (V, E)$

$$f(G) = (V, \underbrace{\{f(e) \mid e \in E\}}_{f(E)})$$

An Isomorphism Between Hypergraphs

v	$1\ 2\ 3\ 4\ 5$
$f(v)$	$1\ 3\ 5\ 2\ 4$



$$E\colon \{1,2\},\{2,3\},\{3,4\},\{4,5\},\{1,2,5\},\{1,2,3,4\}$$

$$f(E)\colon \{1,3\},\{3,5\},\{2,5\},\{2,4\},\{1,3,4\},\{1,2,3,5\}$$

Complexity of GI for Restricted Graph Classes

Deciding graph isomorphism (GI) is in P for restricted graph classes as, e.g.

- **trees** and **planar** graphs [Hopcroft, Tarjan 71]
- graphs of **bounded genus** [Miller 80]
- graphs of **bounded degree** [Luks 82]
- graphs of **bounded eigenvalue multiplicity** [Babai, Grigoryev, Mount 82]

However, the restriction to

- **regular** or **bipartite** graphs

does not decrease the complexity of GI.

Complexity Classes

AM $=$ BP·NP,

NC^i: uniform $\{\wedge, \vee, \neg\}$-circuits of polynomial size and depth $O(\log^i n)$,

TC^i: like NC^i but additionally with threshold gates,

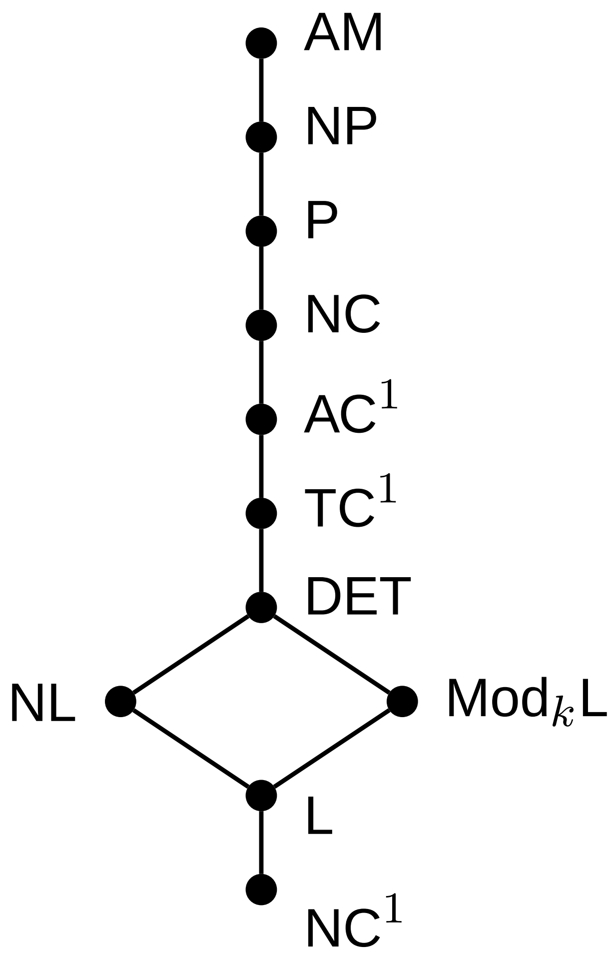
AC^i: like NC^i but with unbounded fanin gates,

NC $= \bigcup_{i \geq 0}$ NC^i,

DET $=$ NC^1(INT-DETERMINANT),

L $=$ DSPACE$(\log n)$, (L = SL [Reingold 04])

NL $=$ NSPACE$(\log n)$,



Mod$_k$L contains all sets A s.t. there is a nondet. logspace machine M with

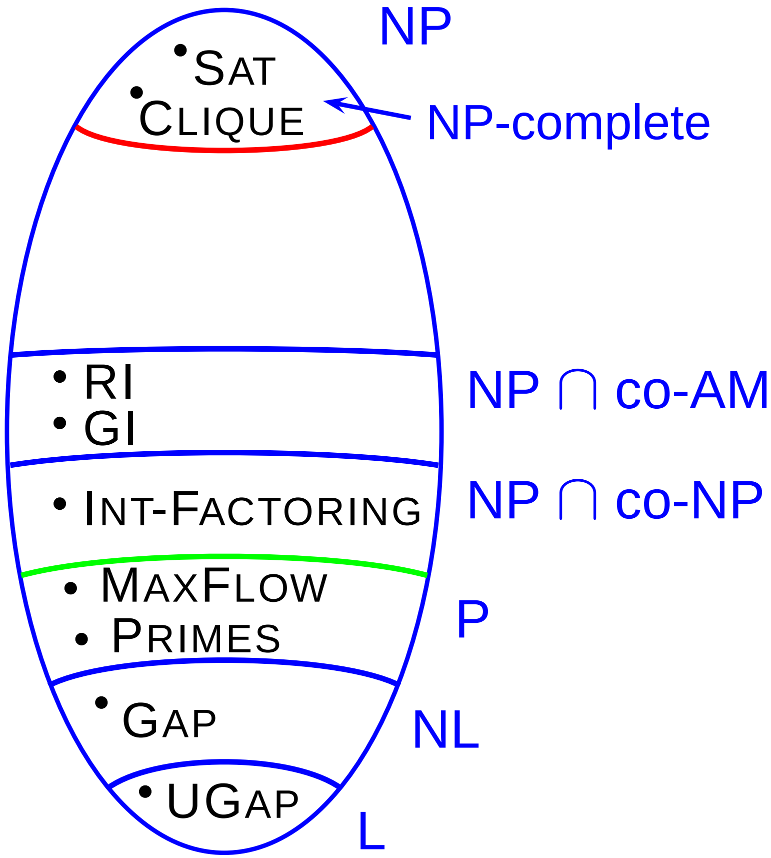
$$A = \{x \mid \#\text{accepting paths of } M(x) \equiv 1 \pmod{k}\}$$

Complexity of GI

- GI $\in$ co-AM [Goldwasser, Sipser 87]
- GI $\in$ SPP [Arvind, Kurur 02]
- GI is hard for DET [Torán 00]
- GI $\in$ NPC $\Rightarrow$ PH $= \Sigma_2^p$

  [Boppana, Hastad, Zachos 87]
- GI $\in$ DTIME$(exp(O(\sqrt{n \log n})))$

  [Luks, Zemlyachenko 83]



Problems Related to GI

GA: Does G have a non-trivial automorphism?

UGI: Is there a unique isomorphism between G and H?

$\#$GI: Determine the number of isomorphisms between G and H.

AUT: Compute a generating set for the automorphism group of G.

ORB: Determine the orbits of the automorphism group of G.

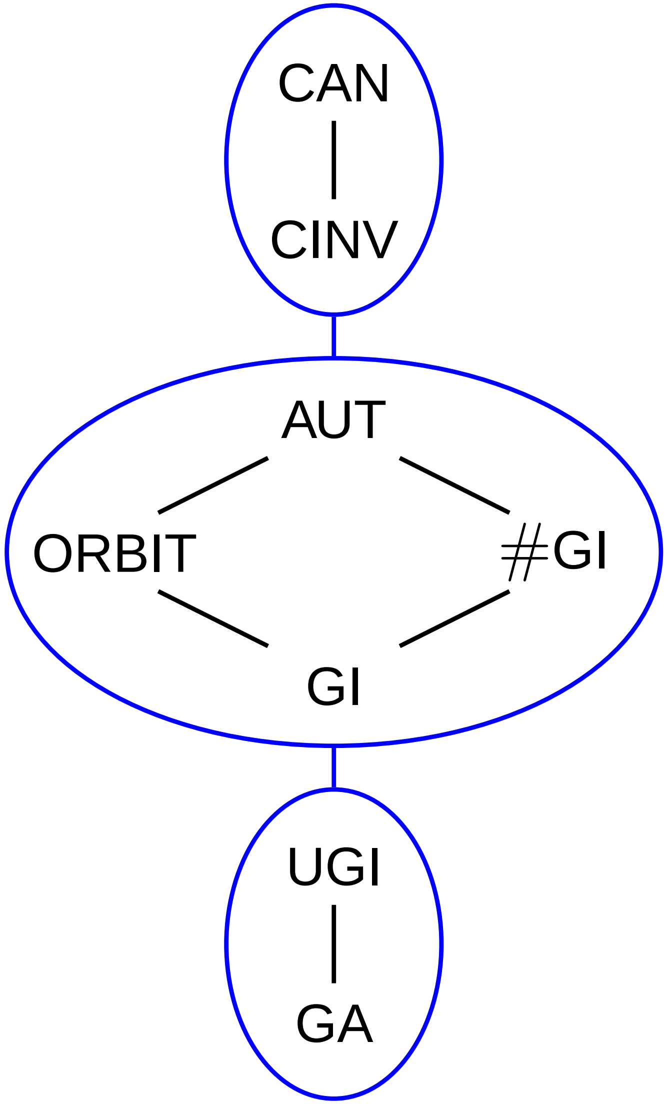
CINV: Compute a complete invariant f for G, i.e.,

for all graphs G and H, $G \cong H \Leftrightarrow f(G) = f(H)$.

CAN: Compute a canonization g for G, i.e.,

for all graphs G and H, $G \cong g(G) \wedge [G \cong H \Rightarrow g(G) = g(H)]$.

Relative Complexity of Problems Related to GI



CAN $\leq_T^P$ CINV [Gurevich 97]

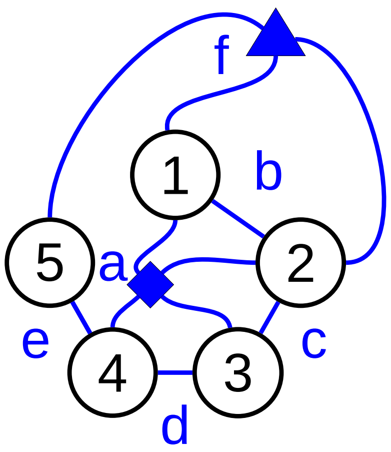
AUT $\leq_T^P$ GI [Mathon 79]

Relative Complexity of GI and HGI

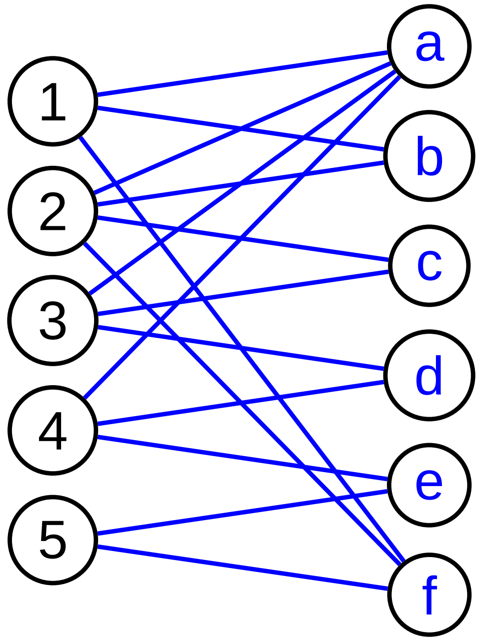
HGI**:** Are hypergraphs G and H isomorphic?

HGA**:** Does hypergraph G have a non-trivial automorphism?

$$G = (V, E) \qquad\qquad G' = (V \cup E, \{\{v, e\} \mid v \in e\})$$



$\mapsto$



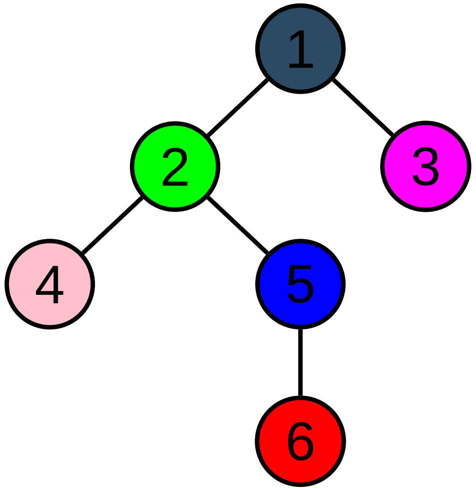
Then we have:

GI $\equiv$ HGI and GA $\equiv$ HGA.

Representing Trees

We consider two different representations of trees:

- by a **string** of nested parentheses,
- by a **pointer list,** i.e., a sequence of edges (ordered pairs).



string representation:

((( ) (( ))) ( ))

pointer list:

(2,5), (2,4), (1,2), (1,3), (5,6)

Complexity of Tree Isomorphism

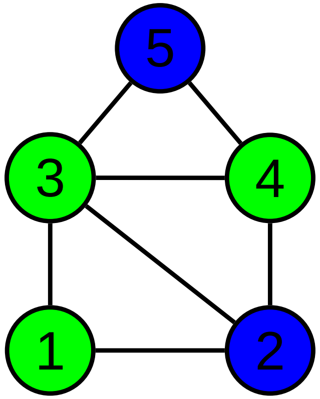
By restricting GI to trees in the string or pointer list representation we obtain the problems STRING-TI and POINTER-TI.

- POINTER-TI $\in$ LINTIME [Aho, Hopcroft, Ullman 74]
- POINTER-TI $\in$ NC [Miller, Reif 91]
- POINTER-TI $\in$ L [Lindell 92] (even canonization)
- STRING-TI $\in$ NC^1 [Buss 97] (even canonization)
- BOUNDED-TREEWIDTH-GI $\in$ TC^1 [Grohe, Verbitsky 06] (even CINV)
- POINTER-TI and STRING-TI are also hard for L and NC^1, respectively [Jenner, K, McKenzie, Torán 03]

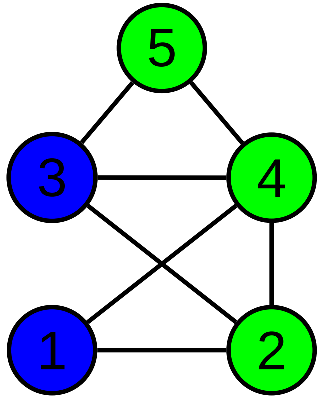
GI for Colored Graphs

COLOR-GI is the problem GI applied to graphs with node colors $c(v)$, which have to be preserved by the isomorphism f,

$$c(f(v)) = c(v).$$



v	1 2 3 4 5
$f(v)$	5 3 4 2 1



Complexity of GI for Bounded Color Classes

By restricting COLOR-GI to graphs having at most k nodes of the same color we obtain the problem COLOR_k-GI.

- $\text{COLOR}_{O(1)}$-GI $\in$ RP [Babai 79]
- $\text{COLOR}_{O(1)}$-GI $\in$ P [Furst,Hopcroft,Luks 80]
- $\text{COLOR}_{O(1)}$-GI $\in \text{NC}^4$ [Luks 86]
- $\text{COLOR}_{O(1)}$-GI $\in \text{NC}^2$ [Arvind, Kurur, Vijayaraghavan 05]

Complexity of Color$_k$-GI for Small k

Color$_2$-HGA
Color$_5$-GI
Color$_5$-GA
Color$_4$-GI
Color$_4$-GA

$\oplus$L-complete
[Arvind, K 06]

Color$_3$-CAN
Color$_2$-CAN
Color$_3$-GI
Color$_2$-GI
Color$_3$-GA
Color$_2$-GA

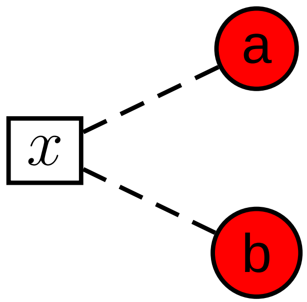
L-complete
[Jenner, K, McKenzie, Torán 03]

Known Upper and Lower Bounds

Problem	hard for	inside
STRING-TA, STRING-CAN	NC^1	NC^1
POINTER-TA, POINTER-CAN	L	L
COLOR_2-GA, COLOR_3-CAN	L	L
COLOR_4-GA, COLOR_5-GI	$\oplus$L	$\oplus$L
COLOR_2-HGA, COLOR_2-HGI	$\oplus$L	$\oplus$L
COLOR_p-HGA, COLOR_p-HGI, p prime	$\text{Mod}_{p!}$L	P
GA, GI	DET	NP $\cap$ co-AM

Warm-Up: $\textsc{Color}_2$-GA $\in$ **L**

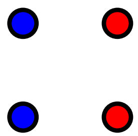
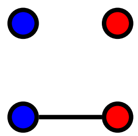
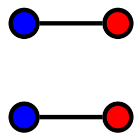
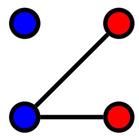
The 1-Bit Representation for Color Classes of Size 2



a	b	x
a	b	0
b	a	1

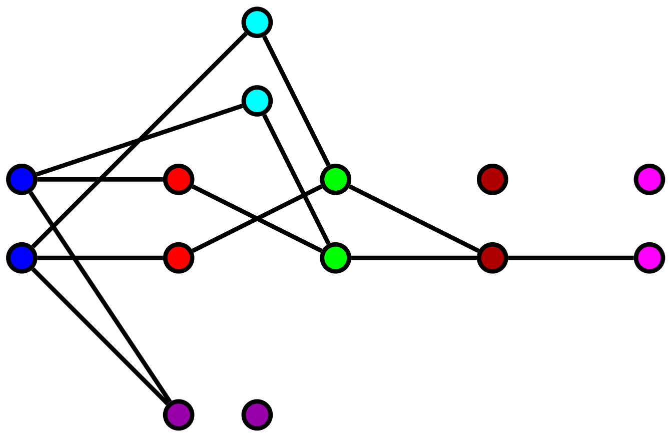
Warm-Up: Color$_2$-GA $\in$ L

Transform graph G with color classes of size 2 into a system S of equations of the form $x_i = x_j$ and $x_i = 0$:

Edges in G between C_i and C_j			
			
-	$x_i = x_j = 0$	$x_i = x_j$	$x_i = 0$
Equations in S			

Then G has a non-trivial automorphism iff S has a non-zero solution.

Warm-Up: Color$_2$-GA $\in$ L



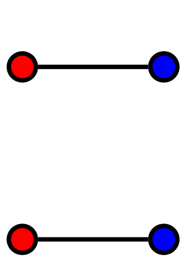
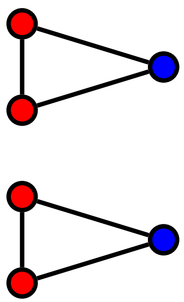
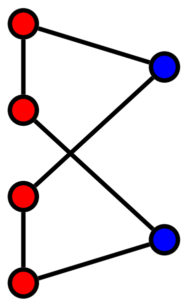
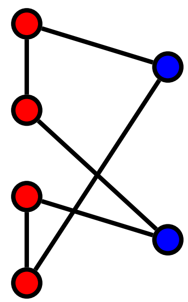
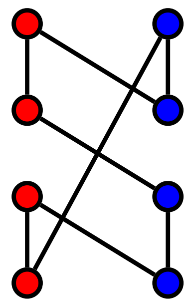
G

$$x_1 = x_2$$
$$x_2 = x_3$$
$$x_3 = x_4$$
$$x_5 = x_6 = 0$$
$$x_7 = 0$$

S

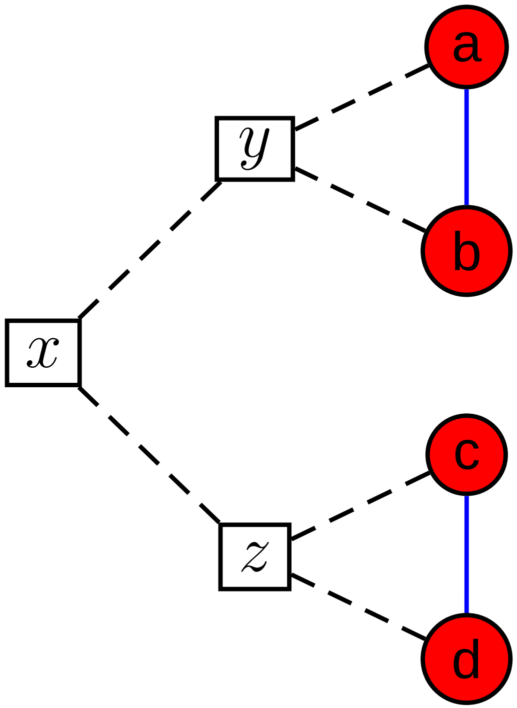
COLOR$_4$-GA $\in \oplus$L

Also COLOR$_4$-GA **can be reduced to solving a system S of equations (this time over $\mathbb{F}_2$:**

Edges in G between C_i and C_j and the corresponding equations				
				
$x_i = x_j$	$x_i = x_j$	$y_i = z_i = x_j$	$x_i = x_j \oplus y_i$, $y_i = z_i$	$x_i = x_j \oplus y_j$, $y_i = z_i = y_j = z_j$

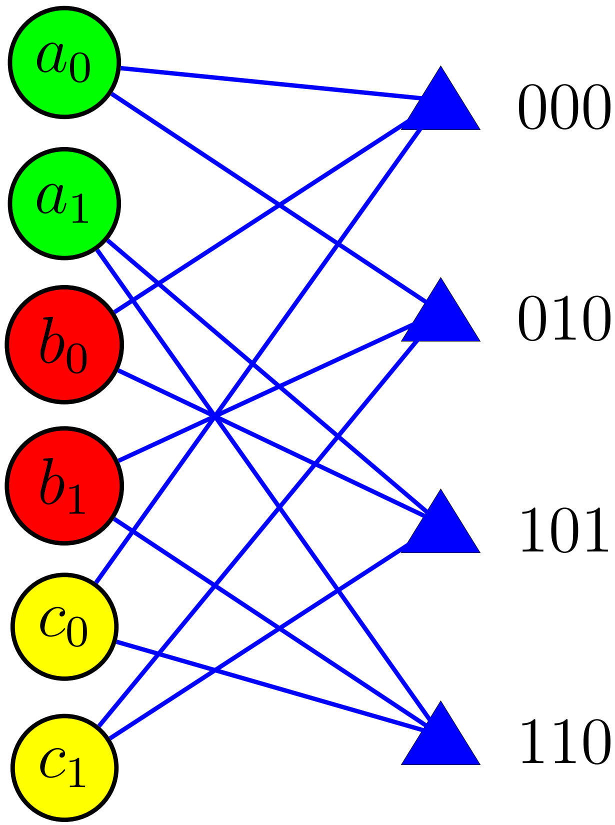
The 3-Bit Representation for Halved Color Classes

A color class of size 4 is called **halved**, if it contains two disjoint edges.



a	b	c	d	x	y	z
a	b	c	d	0	0	0
a	b	d	c	0	0	1
b	a	c	d	0	1	0
b	a	d	c	0	1	1
c	d	a	b	1	0	0
d	c	a	b	1	0	1
c	d	b	a	1	1	0
d	c	b	a	1	1	1

COLOR$_2$-HGA $\in \bigoplus$L



$$x \in Aut(G) \Leftrightarrow \forall e \in E : x \oplus e \in E$$

Open Problems

- Is GI for graphs of bounded treewidth decidable in L?
- Do these graphs admit an NC (or even TC^1) canonization?
- Can we close the gap between the upper and lower complexity bounds for COLOR_k-GI, $k \geq 6$, and for COLOR_k-HGI, $k \geq 3$?
- In particular, are COLOR_k-HGA and COLOR_k-HGI in NC?
- Can Gurevich's reduction of canonization to computing a complete invariant be implemented in NC? (At least for special graph classes?)

THANK YOU