Understanding and Using Spector's Bar Recursion

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A 3-player game

1. Person $i \in \{1, 2, 3\}$ builds a (non-zero) function $g_i(x)$
2. Person i is assigned the number $x_i := g_i(i)$
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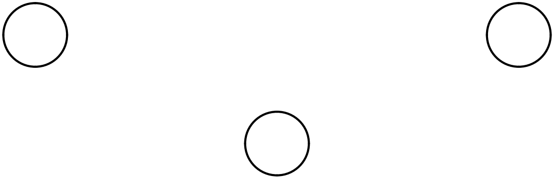

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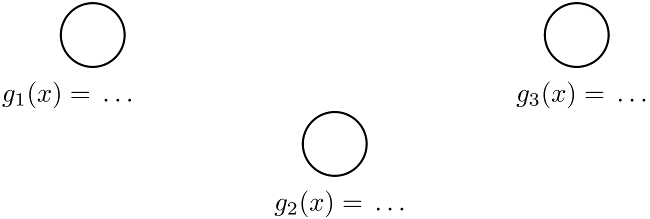

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$g_1(x) = \ldots$

$x_1 := g_1(1)$

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$x_1 := g_1(1)$

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9

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G_3[x_1, x_2] &:= \lambda x_3 . x_1 + x_2 && = \lambda x_3 . 29 \\
X_3[x_1, x_2] &:= x_1 + x_2 && = 29 \\
G_2[x_1] &:= \lambda x_2 . x_1 + X_3[x_1, x_2] && = \lambda x_2 . (x_2 + 18) \\
X_2[x_1] &:= x_1 + X_3[x_1, 2] && = 20 \\
g_1 &:= \lambda x_1 . X_2[x_1] + X_3[x_1, X_2[x_1]] && = \lambda x_1 . (5x_1 + 4) \\
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1. Person $i \in \{1, 2, 3, \ldots\}$ builds a (non-zero) function $g_i(x)$
2. Person i is assigned the number $x_i := \Phi_i(g_i)$
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2. Person i is assigned the number $x_i := \Phi_i(g_i)$
3. $g_i(x_i) = \Delta(x_1, x_2, \ldots)$

Outline

1. Bar recursion
   - Finite bar recursion
   - Spector's bar recursion
2. An application

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- Facts:
  - Classical computational interpretation of countable choice (due to Spector'62)
  - In particular, provides interpretation of full comprehension
  - Difficult to understand
- Goal:
  - Explain bar recursion
  - Use it in simple (practical) examples
  - Understand how it solves the problem

Interpreting countable choice

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- Consider cAC for universal formulas

$$\forall n^{\mathbb{N}} \neg\neg \exists y^{\tau} \forall x^{\sigma} A_{\mathsf{qf}}(n, y, x) \rightarrow \neg\neg \exists f \forall n, x^{\sigma} A_{\mathsf{qf}}(n, fn, x)$$

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Interpretation asks for functionals n, g, f depending on Φ, Ψ, Δ s.t.

$$\neg\neg A_{\mathsf{qf}}(n, \Phi_n g, g(\Phi_n g)) \rightarrow \neg\neg A_{\mathsf{qf}}(\Psi f, f(\Psi f), \Delta f)$$

Interpreting countable choice

How to produce n, g, f (parametrised by Φ, Ψ, Δ) such that

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Enough to satisfy equations:

$$\left\{ \begin{array}{lcl} n & \stackrel{\mathbb{N}}{=} & \Psi f \\ fn & \stackrel{\tau}{=} & \Phi_n g \\ g(fn) & \stackrel{\sigma}{=} & \Delta f \end{array} \right\}$$

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Given $\Psi\hat{\boldsymbol{x}} < |\boldsymbol{x}|$ then $f := \hat{\boldsymbol{x}}$ and $n := \Psi\hat{\boldsymbol{x}}$ and $g := g_n$.

A particular case

Let's consider the particular case in which $\Psi \leq 3$

$$
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$$G_2[x_1] := \lambda x_2.\Delta(x_1, x_2, X_3[x_1, x_2])$$
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Finite bar recursion

General case (with a fixed bound k)

$$\begin{array}{lcl} i & \leq & k \\ x_i & \stackrel{\tau}{=} & \Phi_i g_i \\ g_i(x_i) & \stackrel{\sigma}{=} & \Delta(x_1, \ldots, x_k) \end{array}$$

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General solution can be constructed as follows ($\boldsymbol{x}_{i-1} \equiv x_1, \ldots, x_{i-1}$)

$$\mathsf{fB}(\boldsymbol{x}_{i-1}) = \begin{cases} x_1, \ldots, x_k & k = i - 1 \\ \mathsf{fB}(\boldsymbol{x}_{i-1}, X_i[\boldsymbol{x}_{i-1}]) & \text{otherwise} \end{cases}$$

where $X_i[\boldsymbol{x}_{i-1}] := \Phi_i G_i[\boldsymbol{x}_{i-1}]$ and $G_i[\boldsymbol{x}_{i-1}] := \lambda x_i.\Delta(\mathsf{fB}(\boldsymbol{x}_{i-1}, x_i))$.

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Then take $\langle x_1, \ldots, x_k \rangle := \mathsf{fB}(\langle\ \rangle)$ and $g_i := G_i[\boldsymbol{x}_{i-1}]$.

Spector's bar recursion

Back to the original problem

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\begin{aligned}
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Spector's bar recursion

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can be solved with $(\boldsymbol{x}_{i-1} \equiv x_1, \ldots, x_{i-1})$

$$\mathsf{BR}(\boldsymbol{x}_{i-1}) = \begin{cases} \boldsymbol{x}_{i-1} & \Psi\hat{\boldsymbol{x}} < i - 1 \\ \mathsf{BR}(\boldsymbol{x}_{i-1}, X_i[\boldsymbol{x}_{i-1}]) & \text{otherwise} \end{cases}$$

where $X_i[\boldsymbol{x}_{i-1}] := \Phi_i G_i[\boldsymbol{x}_{i-1}]$ and $G_i[\boldsymbol{x}_{i-1}] := \lambda x_i.\Delta(\mathsf{BR}(\boldsymbol{x}_{i-1}, x_i))$.

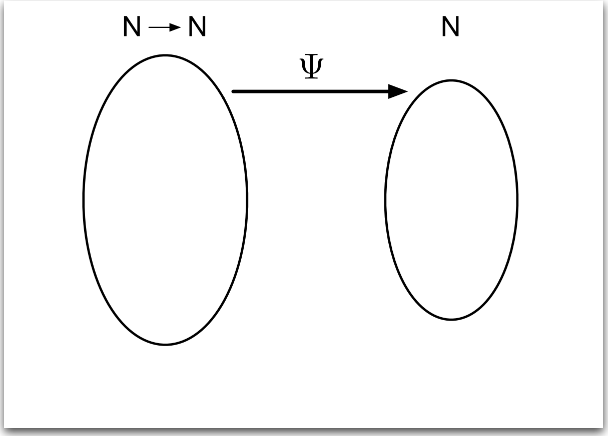
Finally, take $\boldsymbol{x} := \mathsf{fB}(\langle\ \rangle)$ and $g_i := G_i[\boldsymbol{x}_{i-1}]$.

Outline

No injection from $\mathbb{N} \to \mathbb{N}$ to $\mathbb{N}$

Theorem

$\forall \Psi^{1\to 0} \exists \alpha^1, \beta^1 (\alpha \neq \beta \wedge \Psi\alpha = \Psi\beta)$.



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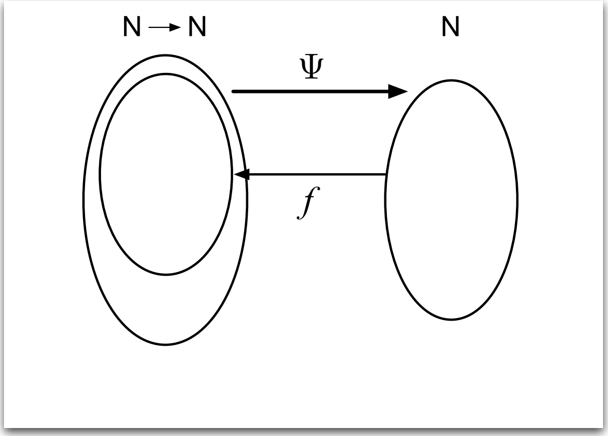

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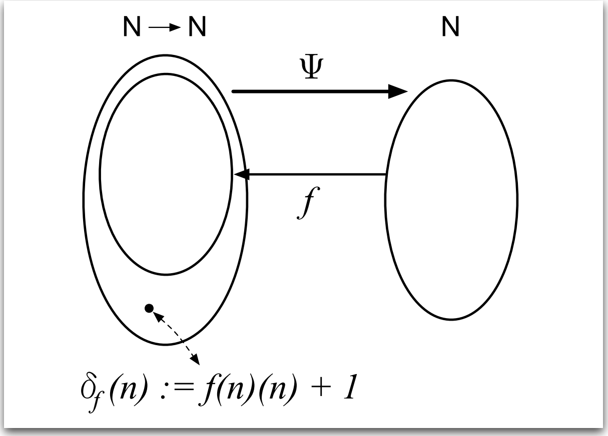

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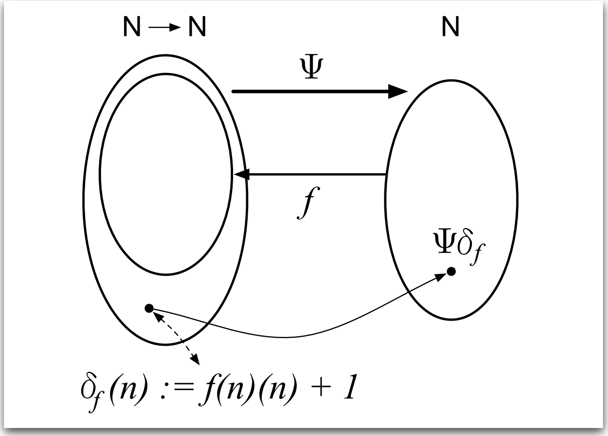

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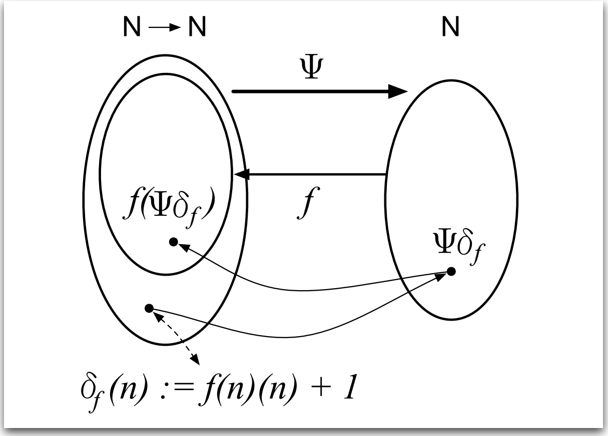

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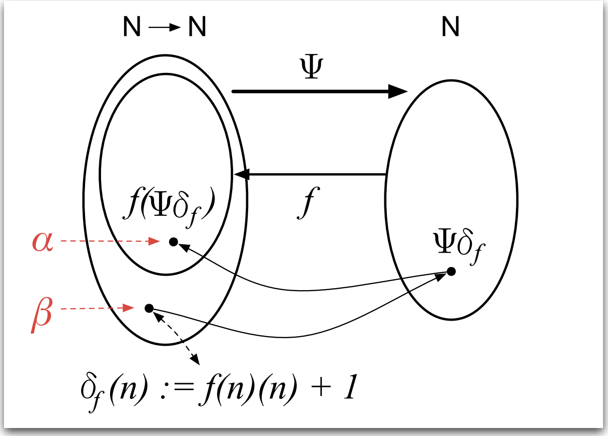

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Let

$$\mathsf{B}(s, k) := \begin{cases} s & \Psi\delta_{\hat{s}} < k \\ r & \Psi\delta_{\hat{r}} \neq k \quad (\text{and } \Psi\delta_{\hat{s}} \geq k) \\ \mathsf{B}(s * \delta_{\hat{r}}, k+1) & \Psi\delta_{\hat{r}} = k \quad (\text{and } \Psi\delta_{\hat{s}} \geq k) \end{cases}$$

where $r := \mathsf{B}(s * 0^1, k+1)$. Then take $t := \mathsf{B}(\langle\, \rangle, 0)$.

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  - Using realizability interpretations:
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