Mathematical Properties of a Class of Input Resolution Refutations

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Mathematical Models of Propositional Proof Systems

Aim: Given a propositional logic $\mathcal{P}$ with axiom system $(\mathcal{A})$. Find mathematical models that represent proofs.

In order to achieve this, we need to establish:

- Which information is sufficient to reconstruct a proof.
- Use mathematical quantities that equivalently describe this information.
- For certain proof systems such as resolution calculus discrete mathematical models can be sufficient to characterise all information contained in a proof.
- We can construct algorithms based on such models to check whether certain proofs exist.
- Mathematical models of proofs may shed some light on questions from complexity theory such as whether or not $\mathcal{NP} = co\mathcal{NP}$ [2,4].

Propositional Resolution Calculus [5,6]

Given: A set of propositional variables v_i, $1 \leq i \leq n$. A clause is a disjunction of literals (a literal is v_i or $\neg v_i$).

Resolution Rule:

For two parent clauses $F \vee l_i$ and $G \vee \bar{l}_i$ we have

$$\frac{F \vee l_i \ , \ G \vee \bar{l}_i}{F \vee G} \ .$$

Particular resolution strategies:

- Linear resolution (derivation tree has only one branch)

- Input resolution (one parent clause is an initial clause)

- Davis-Putnam resolution (particular type of regular resolution)

- Read-once resolution (all initial clauses and any generated resolvent can be used at most once).

Mathematical Models for Resolution

When we mathematically represent resolution proofs, we need to:

- Distinguish resolution proofs with or without tautological resolvents.
- Linear resolution is simpler to be represented than non-linear resolution.
- Input resolution refutations with non-tautological resolvents and without reduction are suitable to be represented by vectors in $\mathbb{R}^n$ (details in [3]).
- Resolution derivations with reduction:

  $$\frac{F \vee G \vee l_i \; , \; F \vee H \vee \bar{l}_i}{F \vee G \vee H}$$

  are considerably more difficult to represent using vectors and may need to be expressed in a different mathematical language.

Comment In [1] a correspondence between propositional resolution proofs and diagrams in the theory of small cancellation is established.

A Mathematical Representation of Clauses

Mathematical models based on sets of vectors and the Euclidean scalar product in $\mathbb{R}^n$ can be used to represent certain propositional resolution proofs.

For a finite propositional language with $v_1, v_2, \ldots, v_n$, each non-tautological clause F creates an equivalence class $[F]$ of clauses containing F, $F \vee l_i \vee \bar{l}_i$ etc.

The non-tautological part of any representant $F \vee l_i \vee \bar{l}_i \ldots$ is identified with a vector in $\mathbb{R}^n$:

$$\vec{F} = \mu_1 \vec{e}_1 + \mu_2 \vec{e}_2 + \ldots + \mu_n \vec{e}_n \ , \quad \mu_i \in \mathbb{Z} \ ,$$

($\mu_i \in \{0, \pm 1\}$ for reduced clauses).

Some Results for Input Resolution

Given a set S of reduced, non-tautological initial clauses. For proofs without reduction we find (for details see [3]):

- If S admits an input resolution refutation of length m it can be characterised by the total length of all initial clauses used in the proof and by the scalar product between them and by their directions (when represented as vectors).
- We introduce a functional $\Phi(S)$ based on scalar products that calculates whether a given set S admits such proofs.
- An algorithm is given which finds all such proofs of length m for a set S (if they exist).

We cannot reproduce similar results for proofs with reduction.

For general resolution refutations weaker results hold and different mathematical methods need to be applied to represent them more suitably.

References

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