

Extensions of the Semi-lattice of the Enumeration Degrees

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Uniform Reducibility: Motivation

Fix a recursive ordinal $0 < \zeta < \omega_1^{CK}$. Denote by \mathcal{S}_ζ the set of all sequences of sets of natural numbers of length ζ .

A natural notion of reducibility $\mathbf{R} \subseteq \mathcal{S}_\zeta^2$ should satisfy at least:

- \mathbf{R} is reflexive and transitive and induces a degree structure $\mathcal{D}_{\mathbf{R}}$.
- \mathbf{R} is notational invariant, i.e. it does not depend on a particular notation of ζ in \mathcal{O} .
- \mathbf{R} is consistent with the enumeration reducibility, i.e. $\mathcal{D}_e \subseteq \mathcal{D}_{\mathbf{R}}$.
- The resulting degree structure $\mathcal{D}_{\mathbf{R}}$ is not trivial.

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Uniform reducibility: Definition

Let z be a notation of ζ in \mathcal{O} and for every $\alpha < \zeta$ denote by z_α the unique notation of α in \mathcal{O} s.t. $z_\alpha \leq_o z$.

Consider two elements $\mathcal{A} = \{A_\alpha\}_{\alpha < \zeta}$ and $\mathcal{B} = \{B_\alpha\}_{\alpha < \zeta}$ of \mathcal{S}_ζ .

- Set $\mathcal{A} \leq_z \mathcal{B}$ iff there exists a recursive function h s.t. $(\forall \alpha < \zeta)(A_\alpha = \Phi_{h(z_\alpha)}(B_\alpha))$, where Φ_a denotes the a -th enumeration operator.

The relation " \leq_z " does not satisfy our requirements: it is not notational invariant and the resulting degree structure is not very interesting.

On the other hand,

$$(\forall \mathcal{A}, \mathcal{B} \subseteq \mathbb{N})(\mathcal{A} \leq_e \mathcal{B} \iff \{A^{(z_\alpha)}\}_{\alpha < \zeta} \leq_z \{B^{(z_\alpha)}\}_{\alpha < \zeta}).$$

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Uniform reducibility: Definition

A way to overcome the drawbacks of " \leq_z " is to extend it in Selman's style to a maximal relation " \leq_z^m ".

$$\mathcal{A} \leq_z^m \mathcal{B} \iff (\forall X \subseteq \mathbb{N})(\mathcal{B} \leq_z \{X^{(z_\alpha)}\}_{\alpha < \zeta} \Rightarrow \mathcal{A} \leq_z \{X^{(z_\alpha)}\}_{\alpha < \zeta}).$$

Clearly " \leq_z^m " is maximal among all transitive relations " \leq " satisfying

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Uniform reducibility: The jump sequence

The *jump sequence* $\{\mathcal{P}_{z_\alpha}(\mathcal{B})\}_{\alpha < \zeta}$:

- (i) $\mathcal{P}_{z_0}(\mathcal{B}) = B_0$.
- (ii) If $\alpha = \beta + 1$, then let $\mathcal{P}_{z_\alpha}(\mathcal{B}) = \mathcal{P}_{z_\beta}(\mathcal{B})' \oplus B_\alpha$.
- (iii) If $\alpha = \lim \alpha(p)$, then set $\mathcal{P}_{< z_\alpha}(\mathcal{B}) = \{\langle p, x \rangle : x \in \mathcal{P}_{z_{\alpha(p)}}(\mathcal{B})\}$ and let $\mathcal{P}_{z_\alpha}(\mathcal{B}) = \mathcal{P}_{< z_\alpha}(\mathcal{B}) \oplus B_\alpha$.

Soskov and Kovachev, 2006:

$$A \leq_z^m B \iff A \leq_z \{\mathcal{P}_{z_\alpha}(\mathcal{B})\}_{\alpha < \zeta}.$$

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The relation " \leq_z^m " satisfies two of the preliminary conditions: It is reflexive and transitive and notational invariant:

$$(\forall z_1, z_2 \in \mathcal{O})(|z_1| = |z_2| = \zeta \Rightarrow (\mathcal{A} \leq_{z_1}^m \mathcal{B} \iff \mathcal{A} \leq_{z_2}^m \mathcal{B})).$$

- (i) $\mathcal{A} \leq_u \mathcal{B} \iff (\exists z \in \mathcal{O})(|z| = \zeta \ \& \ \mathcal{A} \leq_z^m \mathcal{B})$.
- (ii) $\mathcal{A} \equiv_u \mathcal{B} \iff \mathcal{A} \leq_u \mathcal{B} \ \& \ \mathcal{B} \leq_u \mathcal{A}$.
- (iii) Set $d_\zeta(\mathcal{A}) = \{\mathcal{B} \in \mathcal{S}_\zeta : \mathcal{A} \equiv_u \mathcal{B}\}$.
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$\mathcal{D}_\zeta = (\{d_\zeta(\mathcal{A}) : \mathcal{A} \in \mathcal{S}_\zeta\}; \leq_\zeta)$ - the ζ -enumeration degrees.

The element $d_\zeta(\emptyset_\zeta)$ is the least element of \mathcal{D}_ζ and $d_\zeta(\mathcal{A} \oplus \mathcal{B})$ is the least upper bound of $d_\zeta(\mathcal{A})$ and $d_\zeta(\mathcal{B})$.

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Embeddings

- $\mathcal{D}_1 = \mathcal{D}_e$, where \mathcal{D}_e denotes the ordering of the enumeration degrees.
- Let $\alpha < \zeta$ and $\mathcal{A} \in \mathcal{S}_\alpha$. Define the extension $\mathcal{A} \uparrow \zeta \in \mathcal{S}_\zeta$ by:

$$(\mathcal{A} \uparrow \zeta)_\gamma = \begin{cases} \mathcal{A}_\gamma, & \text{if } \gamma < \alpha, \\ \emptyset, & \text{if } \alpha \leq \gamma < \zeta. \end{cases}$$

- For $\mathcal{A}, \mathcal{B} \in \mathcal{S}_\alpha$, $\mathcal{A} \leq_u \mathcal{B} \iff \mathcal{A} \uparrow \zeta \leq_u \mathcal{B} \uparrow \zeta$.
- Let $\kappa(d_\alpha(\mathcal{A})) = d_\zeta(\mathcal{A} \uparrow \zeta)$.
 - $\kappa(\mathbf{0}_\alpha) = \mathbf{0}_\zeta$,
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Embeddings

- The structures \mathcal{D}_α , $\alpha < \zeta$ are *not elementary substructures* of \mathcal{D}_ζ .

Theorem

There exists \mathbf{a} and \mathbf{b} in \mathcal{D}_1 s.t.

- 1 $(\forall)(\alpha < \zeta)(\mathcal{D}_\alpha \models (\exists x)(x = \mathbf{a} \cap \mathbf{b}))$ but
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Let $\alpha < \zeta$ and $\mathbf{a}, \mathbf{b} \in \mathcal{D}_\alpha$. Then

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Embeddings

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Other Embeddings

Definition (Selman, 1971)

Let $n \geq 1$, $A, B \subseteq \mathbb{N}$. Then

$$A \mathfrak{G}_n B \iff (\forall X \subseteq \mathbb{N})(B \in \Sigma_n^X \Rightarrow A \in \Sigma_n^X).$$

For every $A \subseteq \mathbb{N}$ and $n \geq 1$ denote by \mathcal{A}_n the sequence of length n with first $n - 1$ elements equal to \emptyset and the last element equal to A .

Theorem

For every $A, B \subseteq \mathbb{N}$ and $n \geq 1$, $A \mathfrak{G}_n B \iff \mathcal{A}_n \leq_u \mathcal{B}_n$.

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For $\alpha \geq n$, $\mathcal{D}_{\mathfrak{G}_n} \subseteq \mathcal{D}_\alpha$.

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Consider \mathcal{D}_ζ . Assume \mathcal{D}_α is contained in \mathcal{D}_ζ , $\alpha < \zeta$.

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Let \mathbf{a} and \mathbf{b} be elements of \mathcal{D}_ζ . Then

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For every $\mathbf{a} \in \mathcal{D}_\zeta$ there exists a \mathbf{b} such that:

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A sequence $\{A_\gamma\}_{\alpha < \zeta}$ is called *total* if $\{\mathbb{N} \setminus A_\alpha\}_{\alpha < \zeta} \leq_u \{A_\alpha\}_{\alpha < \zeta}$.
A ζ -degree is called *total* if it contains a total sequence.

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For every $\mathbf{a} \in \mathcal{D}_\zeta$ there exists a \mathbf{b} such that:

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Let I be a countable ideal of ζ -degrees. There exist \mathbf{a} and $\mathbf{b} \in \mathcal{D}_1$ s.t.

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The Jump Operation

Let $\mathcal{A} \in \mathcal{S}_\zeta$. Fix $z \in \mathcal{O}$ s.t. $|z| = \zeta$.

Set $\mathcal{A}'_z = \{\mathcal{P}_{z_\alpha}(\mathcal{A})'\}_{\alpha < \zeta}$.

(1) $(\forall z_1, z_2 \in \mathcal{O})(|z_1| = |z_2| = \zeta \Rightarrow \mathcal{A}'_{z_1} \equiv_u \mathcal{A}'_{z_2})$.

(2) $\mathcal{A} <_u \mathcal{A}'_z$.

(3) $\mathcal{A} \leq_u \mathcal{B} \Rightarrow \mathcal{A}'_z \leq_u \mathcal{B}'_z$.

For every $\mathbf{a} = d_\zeta(\mathcal{A}) \in \mathcal{D}_\zeta$ set $\mathbf{a}' = d_\zeta(\mathcal{A}'_z)$.

Theorem (Jump Inversion)

Suppose $\mathcal{B} \in \mathcal{S}_\zeta$ and let \mathcal{C} be a total sequence such that $\mathcal{B}'_z \leq_u \mathcal{C}$. There exists a total sequence \mathcal{F} satisfying the following conditions:

(1) $\mathcal{B} \leq_u \mathcal{F}$.

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Density of the Σ_2^0, ω -degrees

$B \in \mathcal{S}_\omega$ is Σ_2^0 if $B \leq_u \emptyset'_\omega$, i.e. $B \leq_\omega \{\emptyset^{(n+1)}\}_{n < \omega}$.

A combination of Cooper-Gutteridge technique and the Recursion Theorem gives:

Theorem (Density)

Suppose that $A <_u B$ are Σ_2^0 elements of \mathcal{S}_ω . There exists a $C \in \mathcal{S}_\omega$ s.t. $A <_u C <_u B$.

Theorem

Every minimal degree in \mathcal{D}_ω contains a Σ_2^0 sequence.

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There is no minimal degree in \mathcal{D}_ω .

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