Extensions of the Semi-lattice of the Enumeration Degrees

Ivan N. Soskov

Faculty of Mathematics and Computer Science
Sofia University

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Uniform Reducibility: Motivation

Fix a recursive ordinal $0 < \zeta < \omega_1^{CK}$. Denote by $\mathcal{S}_\zeta$ the set of all sequences of sets of natural numbers of length ζ.

A natural notion of reducibility $\mathbf{R} \subseteq \mathcal{S}_\zeta^2$ should satisfy at least:

- $\mathbf{R}$ is reflexive and transitive and induces a degree structure $\mathcal{D}_\mathbf{R}$.
- $\mathbf{R}$ is notational invariant, i.e. it does not depend on a particular notation of ζ in $\mathcal{O}$.
- $\mathbf{R}$ is consistent with the enumeration reducibility, i.e. $\mathcal{D}_e \subseteq \mathcal{D}_\mathbf{R}$.
- The resulting degree structure $\mathcal{D}_\mathbf{R}$ is not trivial.

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Uniform reducibility: Definition

Let z be a notation of ζ in $\mathcal{O}$ and for every $\alpha < \zeta$ denote by z_α the unique notation of α in $\mathcal{O}$ s.t. $z_\alpha \leq_o z$.

Consider two elements $\mathcal{A} = \{A_\alpha\}_{\alpha<\zeta}$ and $\mathcal{B} = \{B_\alpha\}_{\alpha<\zeta}$ of $\mathcal{S}_\zeta$.

- Set $\mathcal{A} \leq_z \mathcal{B}$ iff there exists a recursive function h s.t. $(\forall\alpha < \zeta)(A_\alpha = \Phi_{h(z_\alpha)}(B_\alpha))$, where Φ_a denotes the a-th enumeration operator.

The relation " $\leq_z$ " does not satisfy our requirements: it is not notational invariant and the resulting degree structure is not very interesting.

On the other hand,

$$(\forall A, B \subseteq \mathbb{N})(A \leq_e B \iff \{A^{(z_\alpha)}\}_{\alpha<\zeta} \leq_z \{B^{(z_\alpha)}\}_{\alpha<\zeta}).$$

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A way to overcome the drawbacks of "$\leq_z$" is to extend it in Selman's style to a maximal relation "$\leq_z^m$".

$$\mathcal{A} \leq_z^m \mathcal{B} \iff (\forall X \subseteq \mathbb{N})(\mathcal{B} \leq_z \{X^{(z_\alpha)}\}_{\alpha<\zeta} \Rightarrow \mathcal{A} \leq_z \{X^{(z_\alpha)}\}_{\alpha<\zeta}).$$

Clearly "$\leq_z^m$" is maximal among all transitive relations "$\leq$" satisfying

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Uniform reducibility: The jump sequence

The *jump sequence* $\{\mathcal{P}_{z_\alpha}(\mathcal{B})\}_{\alpha<\zeta}$:

(i) $\mathcal{P}_{z_0}(\mathcal{B}) = B_0$.

(ii) If $\alpha = \beta + 1$, then let $\mathcal{P}_{z_\alpha}(\mathcal{B}) = \mathcal{P}_{z_\beta}(\mathcal{B})' \oplus B_\alpha$.

(iii) If $\alpha = \lim \alpha(p)$, then set $\mathcal{P}_{<z_\alpha}(\mathcal{B}) = \{\langle p, x\rangle : x \in \mathcal{P}_{z_{\alpha(p)}}(\mathcal{B})\}$ and let $\mathcal{P}_{z\alpha}(\mathcal{B}) = \mathcal{P}_{<z_\alpha}(\mathcal{B}) \oplus B_\alpha$.

Soskov and Kovachev, 2006:

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The relation "$\leq_z^m$" satisfies two of the preliminary conditions: It is reflexive and transitive and notational invariant:

$$(\forall z_1, z_2 \in \mathcal{O})(|z_1| = |z_2| = \zeta \Rightarrow (\mathcal{A} \leq_{z_1}^m \mathcal{B} \iff \mathcal{A} \leq_{z_2}^m \mathcal{B})).$$

(i) $\mathcal{A} \leq_u \mathcal{B} \iff (\exists z \in \mathcal{O})(|z| = \zeta \;\&\; \mathcal{A} \leq_z^m \mathcal{B})$.

(ii) $\mathcal{A} \equiv_u \mathcal{B} \iff \mathcal{A} \leq_u \mathcal{B} \;\&\; \mathcal{B} \leq_u \mathcal{A}$.

(iii) Set $d_\zeta(\mathcal{A}) = \{\mathcal{B} \in \mathcal{S}_\zeta : \mathcal{A} \equiv_u \mathcal{B}\}$.

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$\mathcal{D}_\zeta = (\{d_\zeta(\mathcal{A}) : \mathcal{A} \in \mathcal{S}_\zeta\}; \leq_\zeta)$ - the ζ-enumeration degrees.

The element $d_\zeta(\emptyset_\zeta)$ is the least element of $\mathcal{D}_\zeta$ and $d_\zeta(\mathcal{A} \oplus \mathcal{B})$ is the least upper bound of $d_\zeta(\mathcal{A})$ and $d_\zeta(\mathcal{B})$.

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Embeddings

- $\mathcal{D}_1 = \mathcal{D}_e$, where $\mathcal{D}_e$ denotes the ordering of the enumeration degrees.
- Let $\alpha < \zeta$ and $\mathcal{A} \in \mathcal{S}_\alpha$. Define the extension $\mathcal{A} \uparrow \zeta \in \mathcal{S}_\zeta$ by:

$$(\mathcal{A} \uparrow \zeta)_\gamma = \begin{cases} A_\gamma, & \text{if } \gamma < \alpha, \\ \emptyset, & \text{if } \alpha \leq \gamma < \zeta. \end{cases}$$

- For $\mathcal{A}, \mathcal{B} \in \mathcal{S}_\alpha$, $\mathcal{A} \leq_u \mathcal{B} \iff \mathcal{A} \uparrow \zeta \leq_u \mathcal{B} \uparrow \zeta$.
- Let $\kappa(d_\alpha(\mathcal{A})) = d_\zeta(\mathcal{A} \uparrow \zeta)$.
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Embeddings

- The structures $\mathcal{D}_\alpha, \alpha < \zeta$ are *not elementary substructures* of $\mathcal{D}_\zeta$.

Theorem

There exists **a** *and* **b** *in* $\mathcal{D}_1$ *s.t.*

1. $(\forall)(\alpha < \zeta)(\mathcal{D}_\alpha \models (\exists x)(x = \mathbf{a} \cap \mathbf{b}))$ *but*
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Let $\alpha < \zeta$ *and* $\mathbf{a}, \mathbf{b} \in \mathcal{D}_\alpha$. *Then*

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Embeddings

- The structures $\mathcal{D}_\alpha, \alpha < \zeta$ are *not elementary substructures* of $\mathcal{D}_\zeta$.

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Other Embeddings

Definition (Selman, 1971)

Let $n \geq 1$, $A, B \subseteq \mathbb{N}$. Then

$$A \mathfrak{G}_n B \iff (\forall X \subseteq \mathbb{N})(B \in \Sigma_n^X \Rightarrow A \in \Sigma_n^X).$$

For every $A \subseteq \mathbb{N}$ and $n \geq 1$ denote by $\mathcal{A}_n$ the sequence of length n with first $n-1$ elements equal to $\emptyset$ and the last element equal to A.

Theorem

For every $A, B \subseteq \mathbb{N}$ and $n \geq 1$, $A \mathfrak{G}_n B \iff \mathcal{A}_n \leq_u \mathcal{B}_n$.

Corollary

For $\alpha \geq n$, $\mathcal{D}_{\mathfrak{G}_n} \subseteq \mathcal{D}_\alpha$.

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Consider $\mathcal{D}_\zeta$. Assume $\mathcal{D}_\alpha$ is contained in $\mathcal{D}_\zeta$, $\alpha < \zeta$.

Theorem (Analog of Selman's Theorem)

Let **a** *and* **b** *be elements of* $\mathcal{D}_\zeta$. *Then*

$$\mathbf{a} \leq_\zeta \mathbf{b} \iff (\forall \mathbf{c} \in \mathcal{D}_1)(\mathbf{b} \leq_\zeta \mathbf{c} \Rightarrow \mathbf{a} \leq_\zeta \mathbf{c})$$

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For every $\mathbf{a} \in \mathcal{D}_\zeta$ *there exists a* **b** *such that:*

1. $\mathbf{a} <_\zeta \mathbf{b}$.
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A sequence $\{A_\gamma\}_{\alpha<\zeta}$ is called *total* if $\{\mathbb{N} \setminus A_\alpha\}_{\alpha<\zeta} \leq_u \{A_\alpha\}_{\alpha<\zeta}$. A ζ-degree is called *total* if it contains a total sequence.

Theorem (Quasi-minimality II)

For every $\mathbf{a} \in \mathcal{D}_\zeta$ *there exists a* $\mathbf{b}$ *such that:*

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Theorem (Exact Pairs)

Let I *be a countable ideal of* ζ*-degrees. There exist* $\mathbf{a}$ *and* $\mathbf{b} \in \mathcal{D}_1$ *s.t.*

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The Jump Operation

Let $\mathcal{A} \in \mathcal{S}_\zeta$. Fix $z \in \mathcal{O}$ s.t. $|z| = \zeta$.

Set $\mathcal{A}'_z = \{\mathcal{P}_{z_\alpha}(\mathcal{A})'\}_{\alpha<\zeta}$.

(1) $(\forall z_1, z_2 \in \mathcal{O})(|z_1| = |z_2| = \zeta \Rightarrow \mathcal{A}'_{z_1} \equiv_u \mathcal{A}'_{z_2})$.

(2) $\mathcal{A} <_u \mathcal{A}'_z$.

(3) $\mathcal{A} \leq_u \mathcal{B} \Rightarrow \mathcal{A}'_z \leq_u \mathcal{B}'_z$.

For every $\mathbf{a} = d_\zeta(\mathcal{A}) \in \mathcal{D}_\zeta$ set $\mathbf{a}' = d_\zeta(\mathcal{A}'_z)$.

Theorem (Jump Inversion)

Suppose $\mathcal{B} \in \mathcal{S}_\zeta$ and let $\mathcal{C}$ be a total sequence such that $\mathcal{B}'_z \leq_u \mathcal{C}$. There exists a total sequence $\mathcal{F}$ satisfying the following conditions:

(1) $\mathcal{B} \leq_u \mathcal{F}$.

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Density of the Σ^0_2, ω-degrees

$\mathcal{B} \in \mathcal{S}_\omega$ is Σ^0_2 if $\mathcal{B} \leq_u \emptyset'_\omega$, i.e $\mathcal{B} \leq_\omega \{\emptyset^{(n+1)}\}_{n<\omega}$.

A combination of Cooper-Gutteridge technique and the Recursion Theorem gives:

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Suppose that $\mathcal{A} <_u \mathcal{B}$ are Σ^0_2 elements of $\mathcal{S}_\omega$. There exists a $\mathcal{C} \in \mathcal{S}_\omega$ s.t. $\mathcal{A} <_u \mathcal{C} <_u \mathcal{B}$.

Theorem

Every minimal degree in $\mathcal{D}_\omega$ contains a Σ^0_2 sequence.

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There is no minimal degree in $\mathcal{D}_\omega$.

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