

# Uniformization Problems in the Monadic Theory of Countable Ordinals

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# Overview of the talk

- Uniformization problems.
- MLO.
- Definable uniformization and selection.
- Non-anticipatory uniformization.
- Work in progress.

## Uniformization problems

**Definition.** Suppose  $X, Y$  are sets and  $R \subseteq X \times Y$  a binary relation. We say that a function  $f : X \rightarrow Y$  **uniformizes**  $R$  (or is a **uniformizer** for  $R$ ) iff for every  $x \in \text{proj}_X(R)$ ,  $(x, f(x)) \in R$ .

A **uniformization problem** is a pair  $\langle \mathcal{R}, \mathcal{F} \rangle$ , where  $\mathcal{R}$  is a class of binary relation and  $\mathcal{F}$  a class of functions.

Given such a pair we ask whether every  $R \in \mathcal{R}$  has a uniformizer  $f \in \mathcal{F}$ .

# Second-order Monadic Logic of Order (MLO)

**Syntax.** First-order variables  $x, y, z, \dots$

Second-order variables  $X, Y, Z, \dots$

Binary symbols:  $=, <, \in, \subseteq$ .

Atomic formulas have the form  $x = y, x < y, X \subseteq Y, x \in X$ .

Formulas are built up from the atomic formulas by means of the usual Boolean connectives  $\wedge, \vee, \rightarrow, \neg$ , and the quantifiers  $\exists x, \forall x$  and  $\exists X, \forall X$ .

**Semantics.** For us, an MLO-**structure** is a pair  $\mathcal{A} := \langle A, < \rangle$  where  $<$  is a *linear order* of  $A$ .

First-order variables range over *elements* of  $A$ .

Second-order variables range over *subsets* of  $A$ .

$=$ ,  $\in$  and  $\subseteq$  are given their usual meanings.

## Two uniformization problems associated with MLO

Two uniformization problems associated with MLO have been studied in the literature:

- (1) **Definable** uniformization.
- (2) **Non-anticipatory** uniformization (also called Church uniformization or Church synthesis problem).

## Definable uniformization

Fix a linear order  $\mathcal{A} := \langle A, < \rangle$  and let  $\mathcal{R}$  be the class of binary relations over  $\mathcal{P}(A)$  definable in  $\mathcal{A}$ , and  $\mathcal{F}$  the class of definable functions on  $\mathcal{P}(A)$ .

**Definition.** Let  $\mathcal{A}$  a linear order,  $\varphi(X, Y)$ ,  $\psi(X, Y)$  formulas. We say that  $\psi$  **definably uniformizes**  $\varphi$  in  $\mathcal{A}$  iff  $\psi$  defines in  $\mathcal{A}$  the *graph* of a function that is a uniformizer for the relation that  $\varphi$  defines in  $\mathcal{A}$ .

## Known results

The definable uniformization problem was explored over a wide class of structures. In particular, for ordinals, Lifsches and Shelah have shown:

**Theorem** (Lifsches-Shelah, '98). Let  $\alpha$  be an ordinal. The following are equivalent:

- (1)  $\alpha < \omega^\omega$ .
- (2) Every  $\varphi(X, Y)$  has a definable uniformizer in  $(\alpha, <)$ .

However, Lifsches and Shelah took no interest in decidability and computability issues.

## Selection

**Definition.** Let  $\mathcal{A} := \langle A, < \rangle$  be a linear order,  $\varphi(Y)$ ,  $\psi(Y)$  formulas. We say that  $\psi$  is a **selector** for  $\varphi$  in  $\mathcal{A}$  iff there exist a *unique*  $S \subseteq A$  that satisfies  $\psi$  in  $\mathcal{A}$  and this  $S$  also satisfies  $\varphi$ .

Selection is a special case of definable uniformization: replace  $\varphi(Y)$  with  $X = X \wedge \varphi(Y)$ . Then the former has a selector iff the latter has a definable uniformizer.

## Our results

Even the simple problem of selection "fails" at  $\omega^\omega$ :

**Theorem 1** (a) The formula  $\varphi(Y)$  saying " $Y$  is an unbounded  $\omega$ -sequence" has *no* selector in  $(\omega^\omega, <)$ .

(b) There exists an algorithm that, given a formula  $\varphi(Y)$ , decides whether it has a selector in  $(\omega^\omega, <)$ , and - if so - constructs a selector for it.

Interestingly, (a) is the *only* obstacle to selection at  $\omega^\omega$ :

**Theorem 2** Let  $\varphi(Y)$  be a formula. Then there exists a formula  $\psi(Y, Z)$  such for *any* unbounded  $\omega$ -sequence  $S \subseteq \omega^\omega$ ,  $\psi(Y, S)$  is a selector for  $\varphi(Y)$  in  $(\omega^\omega, <)$ . Moreover, this  $\psi$  can be computed from  $\varphi$ .

Thus, every selection problem at  $\omega^\omega$  is effectively reducible to that of selecting an unbounded  $\omega$ -sequence.

The "converse" also holds:

**Theorem 3** For any  $\varphi(Y)$ , we can compute a  $\psi(Y, Z)$  such that the following are equivalent:

(1)  $\varphi(Y)$  has no selector over  $(\omega^\omega, <)$ ;

(2) for *any*  $P \subseteq \omega^\omega$  such that  $(\omega^\omega, <) \models \varphi(P)$ ,  $\psi(Y, P)$  selects an unbounded  $\omega$ -sequence in  $\omega^\omega$ .

## Non-anticipatory functions

**Definition.** Let  $\mathcal{A} := \langle A, < \rangle$  be a linear order,  $f : \mathcal{P}(A) \rightarrow \mathcal{P}(A)$ . We say that  $f$  is **non-anticipatory** iff for all  $X, Y \subseteq A$  and  $a \in A$ , if

$$X \cap (-\infty, a) = Y \cap (-\infty, a),$$

then

$$f(X) \cap (-\infty, a) = f(Y) \cap (-\infty, a)$$

That is, if  $X$  and  $Y$  agree "up to"  $a$ , then so do  $f(X)$  and  $f(Y)$ .

## Non-anticipatory uniformization (a.k.a. Church uniformization)

Fix a linear order  $\mathcal{A} := \langle A, < \rangle$ . Let  $\mathcal{R}$  be the class of binary relations on  $\mathcal{P}(A)$  *definable* in  $\mathcal{A}$  and  $\mathcal{F}$  the class of *non-anticipatory* functions on  $\mathcal{P}(A)$ .

**Fact.** The non-anticipatory uniformization problem "behaves" very differently from the definable uniformization problem.

**Example.** The formula  $\varphi(X, Y)$  that says: "if  $X$  is empty, then  $Y = \text{All}$ ; if  $X$  isn't empty, then  $Y$  is empty" has *no* non-anticipatory uniformizer over any ordinal greater or equal to 2:

Suppose  $X \subseteq \alpha$  and  $0 \notin X$ . If  $0 \in Y$ , then  $\neg\varphi(\{1\}, Y)$ .  
If  $0 \notin Y$ , then  $\neg\varphi(\emptyset, Y)$ .

However,  $\varphi(X, Y)$  can be definably uniformized over any ordinal. In fact, it is *its own* uniformizer.

In 1963, Church turned non-anticipatory uniformization into a decision problem:

**Question** (Church, '63). Does there exist an algorithm that, given a formula  $\varphi(X, Y)$ , determines whether it has a non-anticipatory uniformizer over  $(\omega, <)$ ?

In 1969, Büchi and Landweber answered the question in the affirmative. Their solution is best stated in game-theoretical language.

# Games

Given an ordinal  $\alpha$  and a formula  $\varphi(X, Y)$ , we define a game  $\mathcal{G}_\varphi^\alpha$  as follows.

The game is played by two players called Player  $X$  and Player  $Y$ .

A *play* of the game has  $\alpha$  stages. At stage  $\beta < \alpha$ : first, Player  $X$  chooses  $\pi_X(\beta) \in \{0, 1\}$ ; then, Player  $Y$  chooses  $\pi_Y(\beta) \in \{0, 1\}$ .

At the end of the play  $\pi_X, \pi_Y : \alpha \rightarrow \{0, 1\}$  have been constructed.

Let  $X_\pi := \pi_X^{-1}(1)$ ,  $Y_\pi := \pi_Y^{-1}(1)$ . Then  $Y$  wins the play if

$$(\alpha, <) \models \varphi(X_\pi, Y_\pi);$$

otherwise,  $X$  wins the play.

## Connection with non-anticipatory uniformization

Let  $\alpha$  be an ordinal,  $\varphi(X, Y)$  a formula. Then:

non-anticipatory uniformizer for  $\varphi(X, Y)$  in  $\alpha =$   
winning strategy for Player  $Y$  in the game  $\mathcal{G}_\varphi^\alpha$ .

Thus, we may rephrase Church's question:

**Question.** Does there exist an algorithm that, given a formula  $\varphi(X, Y)$ , decides whether Player  $Y$  has a winning strategy in the game  $\mathcal{G}_\varphi^\omega$ ?

# The Büchi-Landweber theorem

**Theorem** (Büchi-Landweber, '69) Let  $\varphi(X, Y)$  be a formula. Then:

- (a) One of the players has a winning strategy in the game  $\mathcal{G}_\varphi^\omega$ .
- (b) It is decidable *which* of the players has a winning strategy.
- (c) For the player that has a winning strategy, there exists a *definable* (in  $\omega$ ) winning strategy.

(d) There exists an algorithm that, given any  $\varphi(X, Y)$ , constructs a formula  $\psi(X, Y)$  that defines a winning strategy for the winning player in  $\mathcal{G}_\varphi^\omega$ .

## Our results

**Theorem 4** Let  $\alpha$  be a *countable* ordinal,  $\varphi(X, Y)$  a formula.

- (a) One of the players has a winning strategy in the game  $\mathcal{G}_\varphi^\alpha$ .
- (b) It is decidable *which* of the players has a winning strategy.
- (c) If  $\alpha < \omega^\omega$ , then the winning player has a *definable* (in  $\alpha$ ) winning strategy. If  $\alpha \geq \omega^\omega$ , there exists a formula for which this fails.

(d) There exists an algorithm that, given any  $\alpha < \omega^\omega$  and  $\varphi(X, Y)$ , constructs a formula  $\psi(X, Y)$  that defines a winning strategy for the winning player in  $\mathcal{G}_\varphi^\alpha$ .

Recall that Theorem 1 told us that the formula  $\varphi(Y)$  saying " $Y$  is an unbounded  $\omega$ -sequence" has no selector in  $(\omega^\omega, <)$ . Rephrased, it becomes:

**Theorem 1'** Let  $\varphi(X, Y)$  say " $Y$  is unbounded and the order-type of  $Y$  is  $\omega$ ". Then Player  $Y$  wins  $G_\varphi^{\omega^\omega}$ , but there is no *definable* winning strategy for it.

## Work in progress

Recall that Theorem 2 tells us that all formulas  $\varphi(Y)$  are selectable, *if* one allows the use of unbounded  $\omega$ -sequences as parameters. We believe this holds also for the general problem of non-anticipatory uniformization:

**Conjecture.** Let  $\varphi(X, Y)$  be a formula. Then there exists a formula  $\psi(X, Y, Z)$  such that for *any* unbounded  $\omega$ -sequence  $S \subseteq \omega^\omega$ ,  $\psi(X, Y, S)$  defines a winning strategy for the winner of  $\mathcal{G}_\varphi^{\omega^\omega}$ .

Probably, if this conjecture is true, then such a  $\psi(X, Y, Z)$  could be constructed effectively.

**Algorithm wanted 1** Find an algorithm that, given  $\varphi(X, Y)$ , decides whether it has a definable uniformizer over  $\omega^\omega$ , and - if so - constructs one for it.

**Algorithm wanted 2** Find an algorithm that, given  $\varphi(X, Y)$ , decides whether the winner of  $\mathcal{G}_\varphi^{\omega^\omega}$  has a *definable* winning strategy, and - if so - constructs a formula  $\psi(X, Y)$  defining it.

**Thank you.**