On the Real Primitive Recursive Functions and Differential Algebraicity

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Differentially algebraic functions

Definition (E. Moore 1896)

An analytic function $f :\subseteq \mathbf{R} \to \mathbf{R}^n$ is differentially algebraic if for any $x \in \operatorname{dom} f$ we have

$$P\left(f, f', \ldots, f^{(r)}\right) = 0$$

on a neighborhood of x for some $\mathbf{R}$-coefficient nonzero polynomial P.

An analytic function $f :\subseteq \mathbf{R}^m \to \mathbf{R}^n$ is differentially algebraic if it is so as a function of each argument with others held fixed.

Differentially algebraic functions

"Most" analytic functions that mathematicians talk about are differentially algebraic:

- polynomials, rational functions,
- exp, sin, arctan, . . . ,
- their composition, inversion, ...

But some, including the gamma function

$$\Gamma\, x = \int_0^\infty \frac{t^{x-1}}{\exp t}\, \mathrm{d}t \qquad (x > 0),$$

are not (Hölder 1886).

Analog computers

- Continuous (real-valued) state and time
- In practical use during 1930–60s
  before the dominance of digital computers
- A theoretical model:
  the General-Purpose Analog Computer
  (Shannon 1941, Pour-El 1974)
  - Computes unary real functions

Theorem (Lipshitz–Rubel 1987)

The GPAC computes (essentially) all and only the (unary) differentially algebraic functions.

Moore's class

Definition (C. Moore 1996)

Define the class **PR** of real primitive recursive functions by the following closure properties:

Basic functions Constants 0, 1, -1 are in **PR**.

Composition If f and g are in **PR**, so is $f \circ g$.

Juxtaposition If $f_1, \ldots, f_n$ are in **PR**, so is
$$x \mapsto (f_1\, x, \ldots, f_n\, x).$$

Differential recursion If f and g are in **PR**, so is
$\textsc{Dr}\,(f, g)$ (defined next).

Differential recursion

Definition? (C. Moore 1996)

Define $h = \mathrm{D}\textsc{r}\,(f, g)$ by

$$h\,(u, 0) = f\;u$$

$$(\partial/\partial t)h\,(u, t) = g\,\big(u, \tau, h\,(u, \tau)\big),$$

or equivalently by

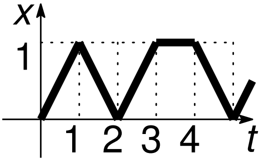
$$h\,(u, t) = f\;u + \int_0^t g\,\big(u, \tau, h\,(u, \tau)\big)\,\mathrm{d}\tau,$$

on the t-interval (containing the origin) on which this equation has a unique solution.

This definition is ambiguous, as we will see.

Moore's claims

- The functions in **PR** are differentially algebraic (and conversely in a sense).
- The real *recursive* functions (obtained by adding another operator) include all arithmetical (in the usual sense) functions (under the embedding $\mathbf{N} \subseteq \mathbf{R}$).



Example use of DR

Before discussing what's wrong with Moore's definition, let us see some examples of the "intended" usage of DR.

Example

From the functions

$$\mathrm{one}\,(\,) = 1, \qquad \mathrm{id}^2_2\,(x, y) = y$$

we get the exponential function by

$$\exp = \mathrm{DR}\,(\mathrm{one}; \mathrm{id}^2_2),$$

which is to say,

$$\exp t = 1 + \int_0^t \exp \tau \,\mathrm{d}\tau.$$

Example use of DR

Example

From the functions

$$\text{zero}\,(\,) = 0, \qquad g\,(x, t) = 1 + t^2$$

we get the tangent function restricted to the interval $(-\pi/2, \pi/2)$ by

$$\tan = \text{D}\textsc{r}\,(\text{zero}; g),$$

which is to say,

$$\tan t = 0 + \int_0^t (1 + \tan^2 \tau)\,\mathrm{d}\tau.$$

Maximal unique solution

$$h(u, t) = f\,u + \int_0^t g\left(u, \tau, h(u, \tau)\right) \mathrm{d}\tau$$

We want to define h on the “biggest possible t-interval on which h uniquely solves the equation,” as we did for tan.

This can be made precise as long as we know what it means for a function to “satisfy” an equation.

But there still remains ambiguity:

Integrand with singularities

Example

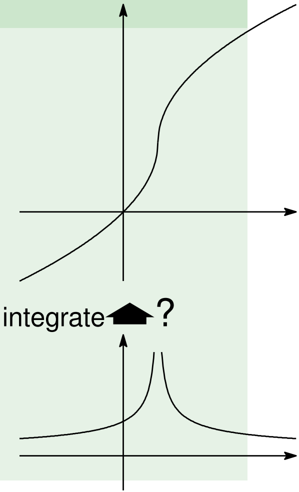
Does

$$h\,t = \begin{cases} 2 + 2 \cdot \sqrt{t - 1} & \text{if } t \geqslant 1 \\ 2 - 2 \cdot \sqrt{-t + 1} & \text{if } t < 1 \end{cases}$$

"satisfy" the equation

$$h\,t = \int_0^t \frac{1}{\sqrt{|\tau - 1|}}\,\mathrm{d}\tau,$$

where the integrand is defined for $\tau \in \mathbf{R} \setminus \{1\}$?



Tolerance to singularities

When is $\int_0^t g\,\tau\,\mathrm{d}\tau$ defined?

Possible answers: It is defined when

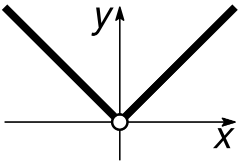
(i) there is a countable set S of isolated points such that g is defined on $[0, t] \setminus S$ and integrable on $[0, t]$ (Campagnolo 2001)

(ii) g is defined and integrable on the whole interval $[0, t]$

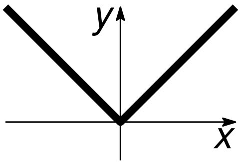
Properties preserved by DR

The following properties of functions are preserved by DR (ii), but not by DR (i):

- Analyticity
- Having open domain
- Being of class C^k (for $k \geq 1$)



In both **PR** (i) (ii)



In **PR** (i) but not in **PR** (ii)

A non-DA function in **PR**

Even if we choose the restrictive DR (ii), Moore's first claim (**PR** $\subseteq$ differentially algebraic) fails:

Proposition

The function

$$f(R, x) = \int_{1/R}^{R} \frac{t^{x-1}}{\exp t} \, \mathrm{d}t \qquad (R, x > 0)$$

is in **PR** but not differentially algebraic.

Proof (non-DA).

For each **Z**-coefficient polynomial P, let $\mathcal{R}_P$ denote the set of all $R > 0$ with

$$P\left(f(R,x), \frac{\partial f(R,x)}{\partial x}, \ldots, \frac{\partial^n f(R,x)}{\partial x^n}\right) = 0 \quad (x > 0).$$

If f is DA, these countably many sets $\mathcal{R}_P$ cover $(0, \infty)$ by Fact 1. Hence, some $\mathcal{R}_P$ has measure > 0. This $\mathcal{R}_P$ must equal $(0, \infty)$ by Fact 2. As $R \to \infty$, the above equation with this P becomes an equation for the gamma function by Fact 3, contradicting Hölder. □

Fact 1 A DA function f satisfies $P(f, f', \ldots, f^{(n)}) = 0$ for some **Z**-coefficient P (Ritt–Gourin 1927).

Fact 2 An analytic function vanishes everywhere if it does so on a set of measure > 0.

Fact 3 $$\frac{d^k \Gamma x}{dx^k} = \lim_{R\to\infty} \frac{\partial^k f(R,x)}{\partial x^k} \qquad (k = 0, 1, \ldots)$$

Open problems

- How can Moore's real *recursive* functions be formalized?

  $$\textsc{Mn}\, f(x, t) = \inf\{\, \tau \geq 0 \mid f(x, \tau) = 0 \,\}$$

  - Should we allow singularities/undefinedness of f?
  - How should $\textsc{Dr}$ work on non-smooth functions (that arise from $\textsc{Mn}$)?
- Can we modify **PR** so that it coincides with the differentially algebraic functions?