

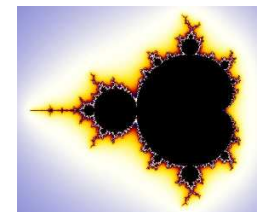
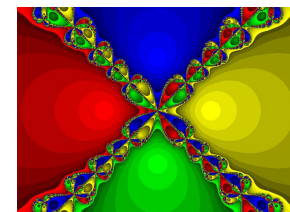
Uncomputability below the real halting problem

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Real Halting Problem

- Halting Problem: *Does a given machine terminate (on empty input)?*
- Discrete H : Turing Machine / Random-Access Machine (TM/RAM)
 - Input/output a finite sequence of bits $\{0, 1\}^*$ / of integers \mathbb{Z}^* .
 - Each memory cell stores one element of $R = \{0, 1\}$ / $R = \mathbb{Z}$.
 - ‘Program’ can store finitely many elements from R ,
 - operates on R (for TM: \vee, \wedge, \neg ; for RAM: $+, -, \times, <$)
- Computation on sorted algebras e.g. [Tucker&Zucker, 1980–]
- Algebra $(\mathbb{R}, +, -, \times, \div, <)$ \rightsquigarrow BCSS-machine on \mathbb{R}^*
e.g. [Blum&Cucker&Shub&Smale 1989–]
 - $\mathbb{H} = \{ \langle M \rangle : \text{BCSS machine } M \text{ terminates on empty input} \} \subseteq \mathbb{R}^*$
 - Undecidable! Also:
 - MANDELBROT Set,
 - $\mathbb{Q} \subseteq \mathbb{R}^*$,
 - Newton Starting Points



I: Word Problem for Groups

- The product bab^2ab^2aba in the free semi-group $\langle\{a, b\}\rangle$, subject to
 - the rules “ $a^4 = a^2, a^3 = b^3, ab = ba$ ”, *cannot* be simplified to 1
 - the additional group rules “ $aa^{-1} = 1 = a^{-1}a$ ” *can* be simplified to 1
- Fix a set X and a set R of equations over $\langle X \rangle = (X \cup X^{-1})^*$.
 - The word problem for $\langle X | R \rangle$ is the following question:
 - Given a formal product $w := x_1^{\pm 1} \cdots x_n^{\pm 1}, x_i \in X$
does it hold, subject to R , “ $w = 1$ ” ?
- For semi-decidable X, R : the word problem is semi-decidable ($\preceq H$)
- There exist finite sets X, R with *undecidable* word problem $\langle X | R \rangle$!
(reducible *from* H) [Boone '58], [Novikov '59]
- But *to a BCSS machine*, any discrete problem becomes decidable:
 - by encoding it into a real number stored as a machine constant.

Word Problem for *Real* Groups

- Let $X \subseteq \mathbb{R}^*$ and a set R of equations over $\langle X \rangle$.
 - Given $w := x_1^{\pm 1} \cdots x_n^{\pm 1}$, does it hold “ $w = 1$ ” (subject to R) ?
 - Caveat: In the free group $\langle \mathbb{R} \rangle$, it is $5^{-1} \neq \frac{1}{5}$!
- E.g. $X = \{x_r : r \in \mathbb{R}\}$, $R = \{x_{nr} = x_r, x_{r+k} = x_r : r \in \mathbb{R}, n \in \mathbb{N}, k \in \mathbb{Z}\}$
 - X, R are BCSS-decidable. It holds: $x_r = 1 \Leftrightarrow r \in \mathbb{Q}$.
 - This word problem is BCSS-*undecidable*! but not BCSS-complete
- For BCSS semi-decidable X, R : the word problem is semi-decidable.
- There are BCSS-decidable X, R with word problem reducible from \mathbb{H}
 - An entirely new kind of BCSS-complete problem!
- For semi-decidable X, R : the word problem is reducible to H .
- There are finite sets X, R with word problem reducible *from* H .

II: Benefit of Real Constants

- BCSS machine may store finitely many *arbitrary* real constants;
 - can decide any *discrete* problem by encoding it into one $c \in \mathbb{R}$.
 - but not the *real* Halting problem \mathbb{H} .
- $\mathbb{H}_d := \{ \langle M \rangle : d\text{-constants BCSS machine } M \text{ terminates on input } 0 \}$
- $H \equiv \mathbb{H}_0 \preceq \mathbb{H}_1 \preceq \mathbb{H}_2 \preceq \dots \preceq \mathbb{H}$ w.r.t. constant-free reduction.
 - Easy. Continuity argument: $\mathbb{Q} \preceq \mathbb{H}_1 \not\preceq_T L$ for any $L \subseteq \mathbb{N}$
 - Intuition: There exists no BCSS-computable *real* pairing function!
 - Theorem:** $\mathbb{H}_{i+1} \not\preceq_m \mathbb{H}_i$ (exploits Sard + Implicit Function Thm)
 - Question: Generalization to Turing reduction?

III: Post's Real Problem

E. POST (1944): “Can an *undecidable r.e. problem* be *strictly easier than (not reducible from) the Halting problem?*”

- FRIEDBERG&MUCHNIK'57/58: Yes, for TMs such a problem *exists!*
 - No explicit/constructive proof known so far.
- [M&Z@FCT'05]: For BCSS machines, it holds $\emptyset \not\preceq \mathbb{Q} \not\preceq \mathbb{H}$!
 - BCSS algorithm can calculate $n \mapsto 2^{2^n}$ (2^n bits!) in $\mathcal{O}(n)$ steps.
 - Program may store (uncomputable) real constants \Rightarrow decide H .
- Linear machine: over $(\mathbb{R}, +, -, 0, 1, <)$;
 - provably related to discrete \mathcal{P} and \mathcal{NP} e.g. [KOIRAN'94]
 - Halting problem \mathbb{H}^ℓ , reduction \preceq^ℓ
- **Theorem:** $\emptyset \not\preceq^\ell \{q^2 : q \in \mathbb{Q}\} \not\preceq^\ell \mathbb{H}^\ell$!

Conclusion

- word problem for real groups is BCSS–complete
- the power of (more) real constants, degree-theoretic investigation
- explicit solution to Post’s problem over the reals, linear BCSS–model



Thank you!