# Gödel, Turing, the Undecidability Results and the Nature of Human Mind

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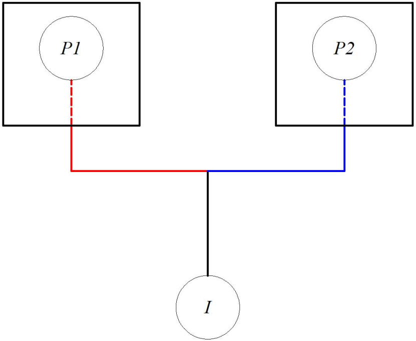
## §1 *Turing and the "Mathematical Objection"*




Turing's Imitation Game (1950) – aka Turing's Test

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3. That would suffice for the Interrogator to make a distinction between the two players.
4. Then, no machine can satisfactorily take the role of a human in the Imitation Game [which, according to Turing's own conventions, *means* that no machine can show an intelligent behaviour].

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*"A human mathematician has always undergone an extensive training ... One must therefore not expect a machine to do a very great deal of building up of instructions table on its own ... [T]he machine must be allowed to have contact with human beings in order that it may adapt itself to their standards."*

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  - character instructions: ending in a peculiar, 'listening' state $\mathbf{q_a}$;
- when the machine is in state $\mathbf{q_a}$, the educator is allowed to either 'reward', or to 'punish' it by means of suitable stimula, causing the instruction just performed either to be adopted definitively, or to be substituted by an alternative (to be chosen by the machine itself among a number of them provided in advance).

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This seems to be presently at the attention of those pursuing directions of research of which Turing's seems an anticipation (i.e., the study of Cognitive and Intelligent Systems by Wiedermann and van Leeuwen).

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Metamathematical investigations (feat. Gödel's theorem) show that not both can be reduced to a mere search of the next most effective step in a derivation according to a finite set of given rules (that is: there's no way to eliminate intuition, and leave only ingenuity).

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"*To convert a brain or machine into a universal machine is the extremest form of discipline. ... But discipline is certainly not enough in itself to produce intelligence. ... Our task is to discover the nature of this residue as it occurs in man, and to try and copy it on machines.*"

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## Turing's philosophical analysis (sort of)

**C.** *Turing's prediction on intelligent machinery: 1950*

*"I believe that in about fifty years' time it will be possible to programme computers ... to make them play the imitation game so well that an average interrogator will not have more than 70 per cent. chance of making the right identification after five minutes of questioning."*

## §2 *Gödel's approach*

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1. Given any suitable and consistent formal system of axioms F, Gödel's so-called second incompleteness theorem in its hypothetical form yields

$$CORR_F \Rightarrow \nvdash_F Con_F$$

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3. Conversely, if a given system of that sort contains all of subjective mathematics, then it cannot be known to be correct with mathematical certitude.

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- A linear or polynomial (quadratic) solution of it in the input $n$, seems "quite within the realm of possibilities".
- If that were really the case, and $n$ is taken to be so large that the restriction to proofs of lenght $n$ can be regarded as irrelevant for any practical purpose, then a human mathematician dealing with questions with an yes-or-no answer could be really substituted by a machine, despite the recursive unsolvability of the general decision problem.

# Is mind mechanical? (2)
## Gödel's "naturalized" Platonism

**A.** *Against constructivism – 1951*

The existence of absolutely unsolvable mathematical problems "*seems to disprove the view that mathematics is only our own creation, for the creator necessarily knows all properties of his creatures because they can't have any other except those he has given to them. So this alternative seems to imply ... some form or other of Platonism or 'realism' as to mathematical objects*".

# Is mind mechanical? (2)
## Gödel's "naturalized" Platonism

**B.** *Knowing the world of mathematical entities – 1951*

The very same nature of mathematical statetment (which are "*true owing to the (nature of the) concepts occurring in it*"), makes it possible for them to be undecidable.

But then our knowledge of the world of concepts would be "*as limited and incomplete as that of the world of things. This occurs in the paradoxes of set theory, which are frequently alleged as a disproof of Platonism, but, I think, quite unjustly.*"

The End