Clocking Type-2 Computation in Unit Cost Model

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Types

– Bertrand Russell (1

Type-0: $\mathbf{N}$ (natural numbers).

$$0, 1, 2, 3 \ldots$$

Type-1: functions, $\mathbf{N} \rightharpoonup \mathbf{N}$.

$$f(x) = x^2, \; g(x) = \frac{1}{3}x^3$$

Type-2: functionals, $(\mathbf{N} \rightharpoonup \mathbf{N}) \rightharpoonup (\mathbf{N} \rightharpoonup \mathbf{N})$.

Mapping from type-1 objects to type-1 objects.

$$(\int x^2) = \frac{1}{3}x^3$$

Type-0 $\subset$ Type-1 $\subset$ Typ

Some more Examples

1. $\Gamma(f) = f \circ f$. $\quad \Gamma : (\mathbf{N} \rightharpoonup \mathbf{N}) \rightharpoonup (\mathbf{N} \rightharpoonup \mathbf{N})$

2. $F(f, x) = f(x)$. $\quad F : (\mathbf{N} \rightharpoonup \mathbf{N}) \times \mathbf{N} \rightharpoonup \mathbf{N}$

3. $G(f, x) = \sum_{i=0}^{x} f(i)$. $\quad G : (\mathbf{N} \rightharpoonup \mathbf{N}) \times \mathbf{N} \rightharpoonup \mathbf{N}$

4. $H(f, x) = \mu i\ [f(i) = x]$. $\quad H : (\mathbf{N} \rightharpoonup \mathbf{N}) \times \mathbf{N} \rightharpoonup \mathbf{N}$

How difficult are they to compute?

Formalisms for computing at typ

- A function ($\mathbf{N} \rightharpoonup \mathbf{N}$) is computable iff it is

  λ-definable (A. Church), or

  recursively definable (K. Gödel/S. Kleene),

  Turing machine computable (A. Turing), o

  you name it!!

- Church-Turing Thesis: All algorithms are computable

While any reasonable formalisms yields the same putable class, a machine model can provide an itive understanding of *computational complexity*

The Foundations of Classical Type-1 Complexity T

$\langle \varphi_i \rangle_{i\in\mathbf{N}}$: an acceptable indexing, $\langle \Phi_i \rangle_{i\in\mathbf{N}}$: a complexity m

1. Complexity Class [**?**]:

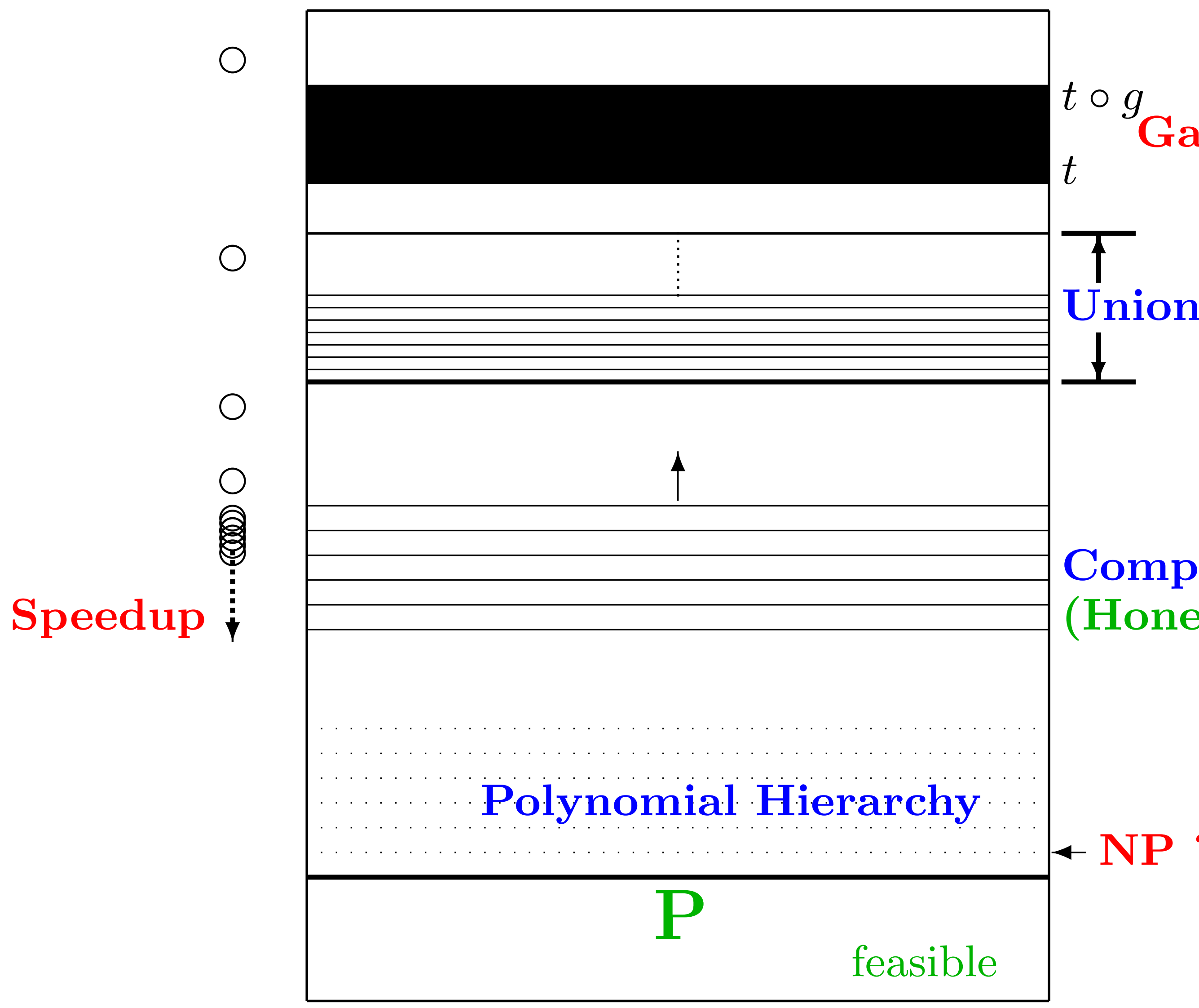
$$\mathbf{C}(t) = \left\{ f \;\middle|\; \exists e \left[\varphi_e = f \wedge \overset{\infty}{\forall}\, x \left(\Phi_e(x) \leq t(|x|) \right)$$

   - $t : \mathbf{N} \to \mathbf{N}$, a computable resource bound.
   - $|x|$, the size of the representation of x, e.g., $|x| \approx$ lo

2. Axiomization (Abstract Complexity Measure [**?**])

   - Axiom 1: $\forall e, x \in \mathbf{N}[\varphi_e(x) \downarrow \iff \Phi_e(x) \downarrow]$.
   - Axiom 2: $\left\{ (e, x, m) \;\middle|\; \Phi_e(x) \leq m \right\}$ is recursive.

Type-1 Complexity Theory



By **Union Theorem** (McCreight and Meyer [**?**]):

- For any computable $f : \mathbf{N} \rightarrow \mathbf{N}$, there is a computabl
  $t : \mathbf{N} \rightarrow \mathbf{N}$ such that

$$\mathbf{C}(t) = O(f), \text{ where } O(f) = \bigcup_{a,b \in \mathbf{N}} \mathbf{C}(af + b)$$

- There is a computable $t : \mathbf{N} \rightarrow \mathbf{N}$ such that,

$$\mathbf{C}(t) = O(1) \cup O(n) \cup O(n^2) \cup O(n^3) \cdots = \mathbf{P}$$

- There is a computable $t : \mathbf{N} \rightarrow \mathbf{N}$ such that,

$$\mathbf{C}(t) = \mathbf{EXP}.$$

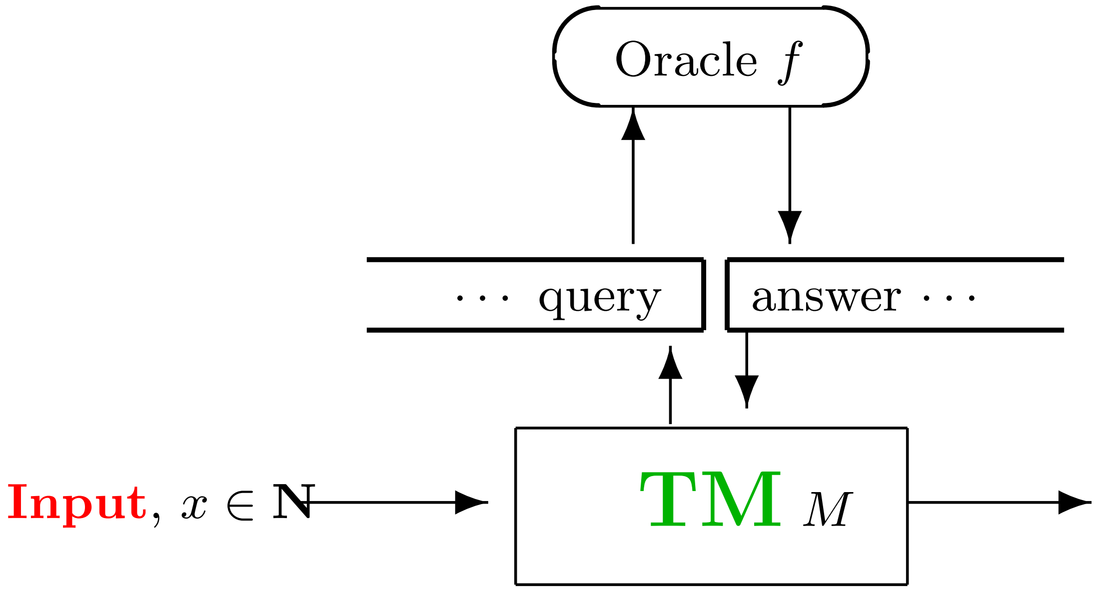
How about Type-2 Computation?

Good News:

Oracle Turing Machine (OTM)

– a widely accepted machine model for type-2 computation.

Oracle Turing Machine [?]



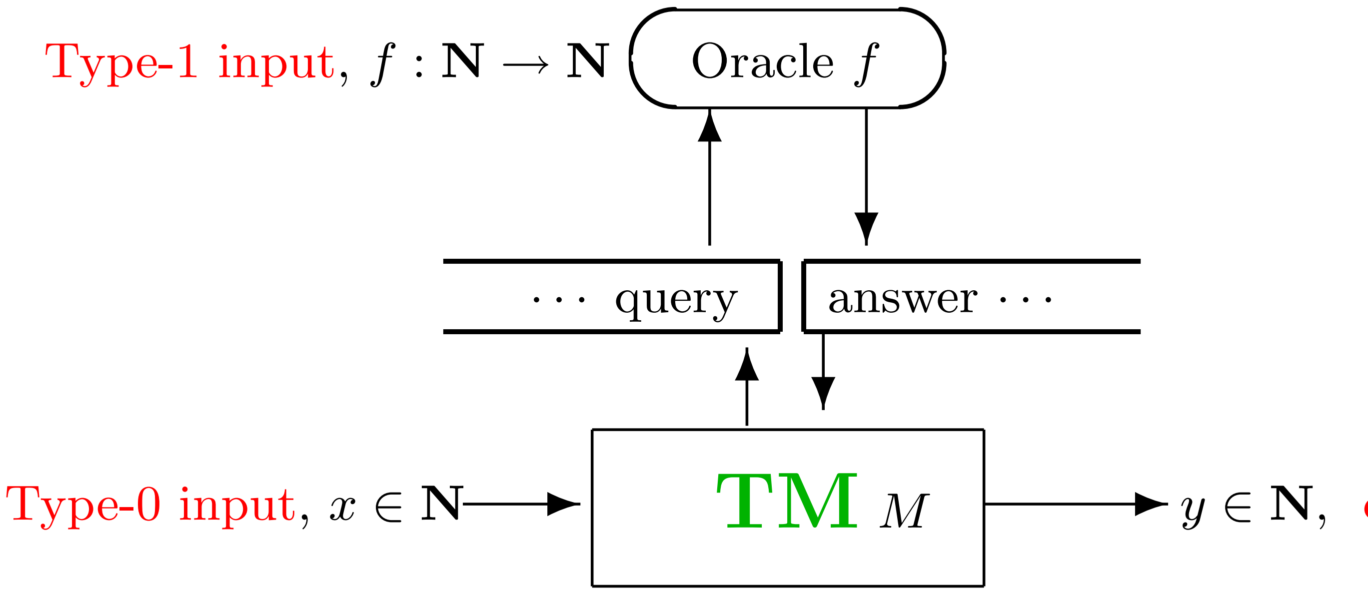
$M^f(x) =$

Cook Reduction [?]

$$M^{\text{SAT}}(x) = \begin{cases} 1 & \text{if } x \in \text{TSP} ; \\ 0 & \text{otherwise,} \end{cases}$$

Computable $F : (\mathbf{N} \to \mathbf{N}) \times \mathbf{N} \rightharpoonup \mathbf{N}$

Oracle Turing Machines (OTMs)



$$F(f, x) = M^f(x) = y$$

Bad News:

- There is no corresponding Church-Turing thesis at typ

  –
```
if f = g then return 1;
        else return 0;
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  How to handle finitely many type-1 inputs?

  How to patch our program on finitely many bad type-

  What should be a workable notion of asymptotic comp

- How **to clock** a machine? **to shut down**?

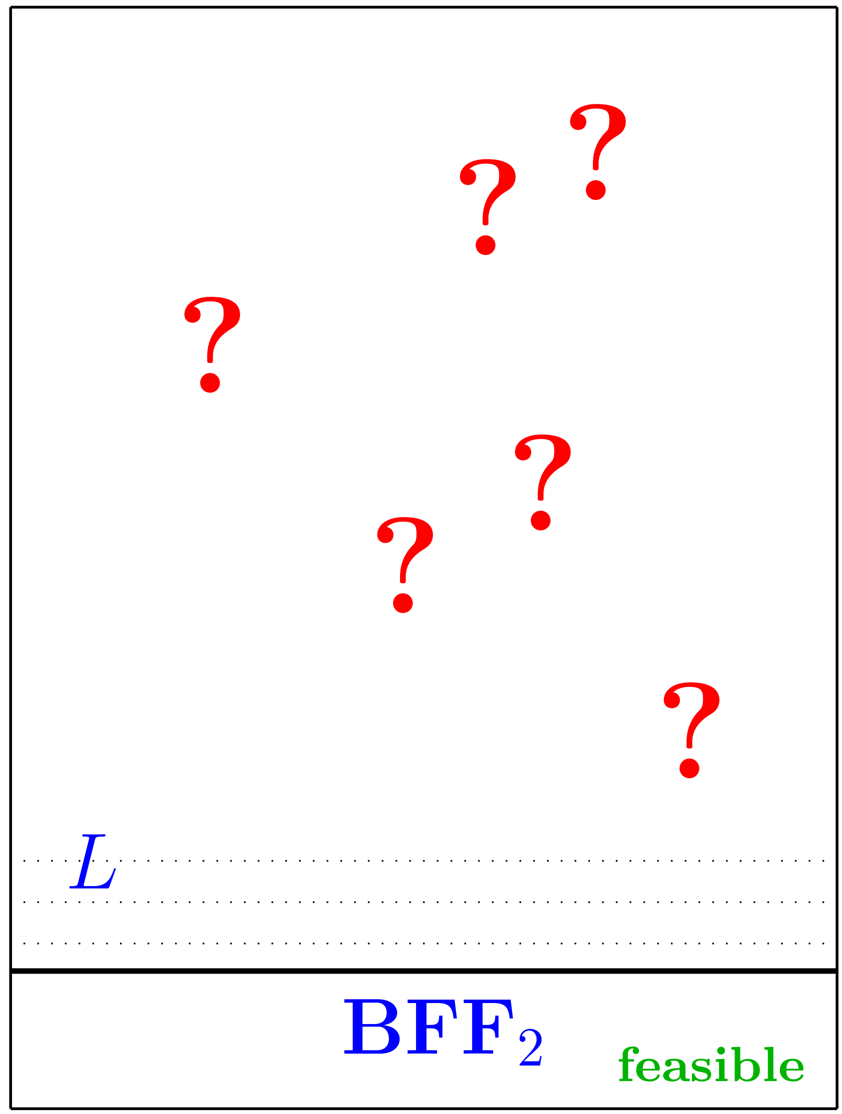
  What is the sort of clocks (bounds) for type-2 comput

  Why POTM (Polynomial-time OTM) [**?**, **?**] isn't so n

Type-2 Complexity Theory

$\mathbf{C}(T) \triangleq ?$

$T = ?$



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Our solutions:

- (Time/Space) constructible bounds (clocks) $\rightarrow$

  Type-2 Time Bounds ($\mathbf{T_2T}$

  $\mathbf{T_2TB}$: $\mathcal{F} \times \mathbf{N} \rightarrow \mathbf{N}$ .... convergent ....

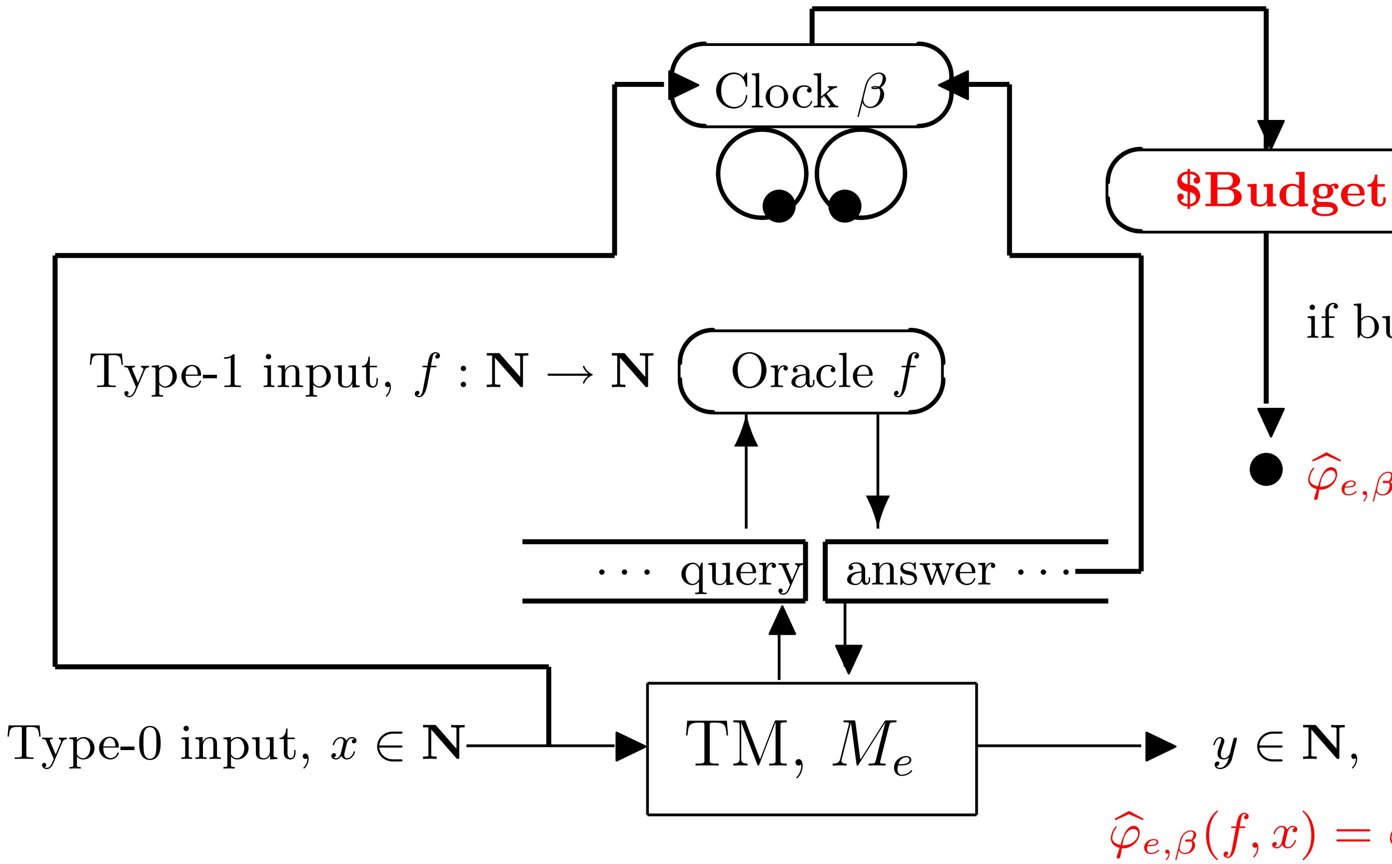
- Finite $\rightarrow$ **compact** in ....

  Let $\beta \in \mathbf{T_2TB}$.

$$\mathbf{C}(\beta) = \left\{ \widehat{\varphi}_e \;\middle|\; E_{e,\beta} \textbf{ is compact} \right\},$$

where $E_{e,\beta} \;\triangleq\; \left\{ (f, x) \;\middle|\; \widehat{\varphi}_{e,\beta}(f, x) \Uparrow \right\}.$

A **Clocked** Oracle Turing Machine



$\widehat{\varphi}_{e,\beta} : (\mathbf{N} \to \mathbf{N}) \times \mathbf{N} \rightharpoonup \mathbf{N}$

$\beta \in \mathbf{T_2TB}$ is **Locking Detectable** if β has a locking det
such that, $\ell : \mathcal{F} \times \mathbf{N} \to \{0, 1\}$ is recursive and for every to
function f, finite function σ, and $x \in \mathbf{N}$, we have

1. $\ell(\sigma, x) = 1 \Longrightarrow \beta(\sigma, x)\downarrow.$

2. $\lim\limits_{\sigma \to f} \ell(\sigma, x) = 1.$

Theorem: There is an effective operator Θ : $\mathbf{T_2TB}$ –
such that, for every $\beta \in \mathbf{T_2TB}$, we have $\mathbf{C}(\beta) = \mathbf{C}(\Theta(\beta)$

Theorem: Basic Hierarchy

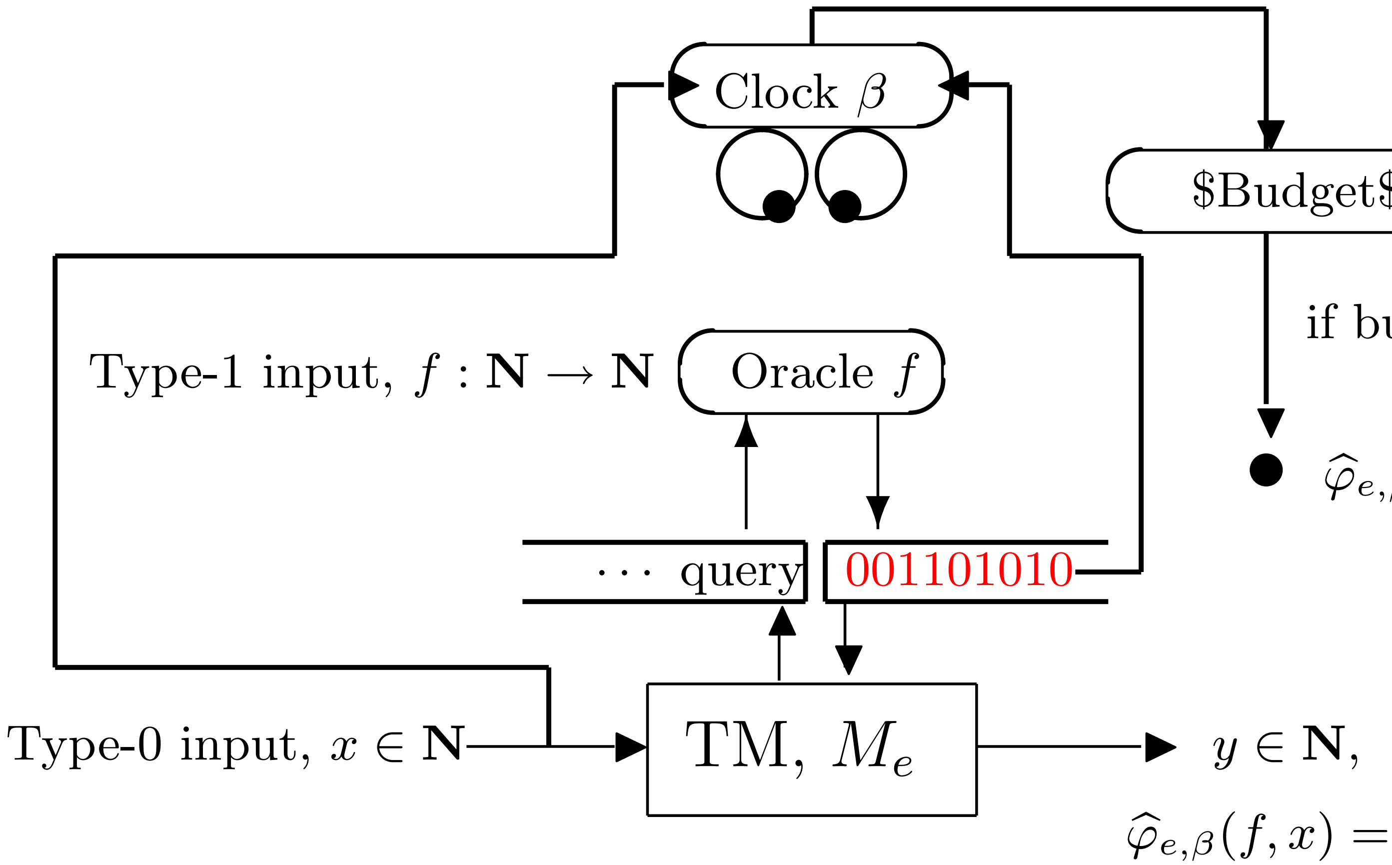
$$\forall \beta_1, \beta_2 \in \mathbf{T}_2\mathbf{TB} \left[\beta_1 \leq \beta_2 \implies \mathbf{C}(\beta_1) \subseteq \mathbf{C}(\beta_2) \right].$$

Question: Is there any effective operator Θ such that
$\beta \in \mathbf{T}_2\mathbf{TB}$, we have

$$\mathbf{C}(\beta) \subset \mathbf{C}(\Theta(\beta)) \subset \mathbf{C}(\Theta^2(\beta)) \subset \mathbf{C}(\Theta^3(\beta)) \subset \cdots\cdots$$

The classical Operator Gap Theorem [**?**, **?**] says **NO** to th
complexity classes.

Answer-Length cost model vs. **Unit** cost mode



$\mathbf{C}_A(\beta)$ **vs.** $\mathbf{C}_U(\beta)$

Theorem: **(Weak Type-2 Gap Theorem)** Given any
increasing recursive function $g : \mathbf{N} \to \mathbf{N}$, there exists β
such that,

$$\mathbf{C}_{\mathbf{A}}(\beta) = \mathbf{C}_{\mathbf{A}}(g \circ \beta).$$

Theorem: **(Anti-Gap Theorem I)** For every $\beta \in \mathbf{T}_2\mathbf{T}$
exists a recursive function $g : \mathbf{N} \to \mathbf{N}$ such that

$$\mathbf{C}_{\mathbf{U}}(\beta) \subset \mathbf{C}_{\mathbf{U}}(g \circ \beta).$$

Theorem: **(Weak Anti-Gap Theorem II)** Suppose g
is recursive and, for every $x \in \mathbf{N}$, $g(x) \geq 3x$. Then,
strong $\beta \in \mathbf{T}_2\mathbf{TB}$,

$$\mathbf{C}_{\mathbf{U}}(\beta) \subset \mathbf{C}_{\mathbf{U}}(g \circ \beta).$$

Theorem: (Inflation Theorem) There is a recursive op
such that, for each $\beta \in \mathbf{T_2TB}$, we have $\Theta(\beta) \in \mathbf{T_2TB}$ an

$$\mathbf{C}(\beta) \subset \mathbf{C}(\Theta(\beta)).$$

Corollary: (Compression Theorem)

$$\mathbf{C}(\beta) \subset \mathbf{C}(\Theta(\beta)) \subset \mathbf{C}(\Theta^2(\beta)) \subset \mathbf{C}(\Theta^3(\beta)) \subset \cdots$$

Hold in both models

Just like at type-1, the union of arbitrary comp ity classes fail to be a complexity class in gene

Theorem: There exist $\beta_1, \beta_2 \in \mathbf{T_2TB}$ such that,

$$\forall \alpha \in \mathbf{T_2TB}, \mathbf{C}(\alpha) \neq \mathbf{C}(\beta_1) \cup \mathbf{C}(\beta_2).$$

Theorem: $\mathbf{BFF_2}$ is not type-2 complexity class, i.e.,

$$\forall \beta \in \mathbf{T_2TB}[\mathbf{C}(\beta) \neq \mathbf{BFF_2}].$$

The Union Theorem (McCreight and Meyer

Theorem: Given any sequence of recursive functions f_0,
such that,

(i) $\lambda i, x . f_i(x)$ is recursive, and

(ii) $f_0 \leq f_1 \leq f_2 \leq \cdots$

there is a recursive function g such that $C(g) = \bigcup_{i \in \mathbf{N}} C(f_i$

The conditions are rather weak.

- P, $PSPACE$, EXP, Big-$O(t)$, ect., are all complexit

Definitions:

Let $\langle\beta_i\rangle$ denote a sequence of type-2 time bounds β_0, β_1, β

1. We say that $\langle\beta_i\rangle$ is ***uniform*** if and only if $\lambda i, \sigma, x.\beta_i(\sigma, x$
recursive.

2. We say that $\langle\beta_i\rangle$ is ***ascending*** if and only if, for all $i \in \mathbf{N}$
$\beta_i \leq \beta_{i+1}$.

3. We say that $\langle\beta_i\rangle$ is ***useful*** if and only if, for all $i \in \mathbf{N}$, β_i

4. We say that $\langle\beta_i\rangle$ is ***convergent*** if and only if, for every
$(f, x) \in \mathcal{T} \times \mathbf{N}$, there is a $\sigma \subset f$ such that, $\beta_i(\sigma, x) \downarrow$ for ev

5. We say that $\langle\beta_i\rangle$ is ***uniformly convergent*** if and only
every $n \in \mathbf{N}$ and $(\sigma, x) \in \mathcal{F} \times \mathbf{N}$, if $\beta_n(\sigma, x) \downarrow$, then for all
$\beta_i(\sigma, x) \downarrow$.

6. We say that $\langle\beta_i\rangle$ is ***strongly convergent*** if and only if
uniformly convergent and there is a recursive locking detec

Under answer-length cost model

Theorem: (Type-2 Union Theorem) Suppose that $\langle$
uniform, (2) ascending, (3) useful, (4) strongly convergen
there is an $\alpha \in \mathbf{T_2TB}$ such that, $\mathbf{C_A}(\alpha) = \mathbf{C_A}(\langle \beta_i \rangle)$.

Type-2 Big-O

Given $\beta \in \mathbf{T_2TB}$, define

$$\mathbf{O}(\beta) = \left\{ \widehat{\varphi}_e \;\middle|\; \widehat{\varphi}_e \in \mathbf{C}(c\beta + d) \text{ for some } c, d \in \mathbf{N} \right\}$$

Under unit cost model

Theorem: For every $\beta \in \mathbf{T_2TB}$, $\mathbf{O}(\beta)$ is not a complex
under unit-cost model.

Conclusion and current and future study

- The two different models give two very different type-
  complexity theories.
- What is the notion of honesty at type-2?
- Speedup theorems?