An Analysis of the Lemmas of Urysohn and Urysohn-Tietze according to effective Borel measurability

Guido Gherardi
Dipartimento di Scienze Mathematiche
e Informatiche R. Magari
Università di Siena

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Introduction to computable analysis
and theory of representations

Notations and Representations.

Given a set S, a notation ν of S is a surjective function $\nu :\subseteq \mathbb{N} \to S$.

Given a set S, a representation γ of S is a surjective function $\gamma :\subseteq \mathbb{N}^{\mathbb{N}} \to S$.

Computable metric space.

The tuple $\mathbf{M} = (M, d, Q, \nu_{Q})$ is a computable metric space if:

- $M \neq \emptyset$ is a set;
- (M, d) is a complete metric space;
- Q is a countable dense subset of M;
- ν_{Q} is a notation of Q with dom$(\nu_{Q}) = \mathbb{N}$;
- the set $e_d = \{(n, m, k, i) \in \mathbb{N}^4 : \nu_{\mathbb{Q}}(n) < d(\nu_{Q}(m), \nu_{Q}(k)) < \nu_{\mathbb{Q}}(i)\}$ is r.e..

$\mathcal{A}$ will be the class of all the closed sets in the topology generated by d.

The set of all balls $B(q, \alpha)$ with $q \in Q$ and $\alpha \in \mathbb{Q}^+$ is a base for the topology generated by d. We call the elements of this set denoted open balls of $\mathbf{M}$, since we denote them by a notation ν_{M} such that dom$(\nu_{\mathrm{M}}) = \mathbb{N}^+$.

The ball $\nu_{\mathrm{M}}(n)$, for $n > 0$, will be denoted by "I_n^{M}".

Standard representation.

δ_{M} is the standard representation of M associated with $\mathbf{M}$: for $p \in \mathbb{N}^{\mathbb{N}}$, let

$$\delta_{\mathrm{M}}(p) = x \in M \Leftrightarrow \{n > 0 : n \in p\} = \{n : x \in I_n^{\mathrm{M}}\}.$$

ρ denotes the standard representation associated with $\mathbb{R}$.

We represent $\mathbb{N}^{\mathbb{N}}$ by the standard representation $\delta_{\mathbb{B}}$ associated with the Baire computable metric space $\mathbb{B}$.

Representation of Borel sets.

Let $\mathbf{M} = (M, d, Q, \nu_{_Q})$ be a computable metric space. Define by induction the following representation of the class of Borel sets generated in M by d:

- $\delta_{\mathbf{\Sigma}^0_1(\mathbf{M})}(p) = \bigcup_{\langle m,k\rangle+1\in p} B(\nu_{_Q}(m), \overline{\overline{k}})$;
- $\delta_{\mathbf{\Pi}^0_k(\mathbf{M})}(p) = M \setminus \delta_{\mathbf{\Sigma}^0_k(\mathbf{M})}(p)$;
- $\delta_{\mathbf{\Sigma}^0_{k+1}(\mathbf{M})}\langle p_0, p_1, p_2, \ldots\rangle = \bigcup_{i=0}^{\infty} \delta_{\mathbf{\Pi}^0_k(\mathbf{M})}(p_i)$;
- $\delta_{\mathbf{\Delta}^0_k(\mathbf{M})}\langle p, q\rangle = \delta_{\mathbf{\Sigma}^0_k(\mathbf{M})}(p) \Leftrightarrow \delta_{\mathbf{\Sigma}^0_k(\mathbf{M})}(p) = \delta_{\mathbf{\Pi}^0_k(\mathbf{M})}(q)$.

Σ^0_k-computability.

Given a function $F :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ we say that F is $\boldsymbol{\Sigma}^0_{k+1}$-*computable* if there is a computable function $G :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ mapping any $\delta_{\boldsymbol{\Sigma}^0_1(\mathbb{N}^\mathbb{N})}$-representation of $O \in \boldsymbol{\Sigma}^0_1(\mathbb{N}^\mathbb{N})$ to some $\delta_{\boldsymbol{\Sigma}^0_{k+1}(\mathbb{N}^\mathbb{N})}$-representation of a $\boldsymbol{\Sigma}^0_{k+1}(\mathbb{N}^\mathbb{N})$-set V such that $F^{-1}(O) = V \cap \mathrm{dom}(F)$.

Let two represented sets $(S_1, \delta_1), (S_0, \delta_0)$ be given and let also a function $f :\subseteq S_1 \to S_0$ be given. We say that f is $\boldsymbol{\Sigma}^0_{k+1}$-computable w.r.t. representations (δ_1, δ_0) if f has a $\boldsymbol{\Sigma}^0_{k+1}$-computable (δ_0, δ_0)-realization F.

$$\begin{tikzcd}
\mathbb{N}^\mathbb{N} \arrow[r, "F"] \arrow[d, "\delta_1"'] & \mathbb{N}^\mathbb{N} \arrow[d, "\delta_0"] \\
S_1 \arrow[r, "f"'] & S_0
\end{tikzcd}$$

Fact 1. The function

$$\lim_{\mathbb{B}} : \langle p_n \rangle_{n \in \mathbb{N}} \mapsto \lim_{n \to \infty} p_n$$

with $\{p_n\}_{n \in \mathbb{N}}$ a convergent sequence in $\mathbb{B}$, is $\mathbf{\Sigma}^0_2$-computable.

Reducibility.

Let represented sets (S_i, δ_i), for $1 \leq i \leq 4$, be given and let functions $f :\subseteq S_1 \to S_2$, $g :\subseteq S_3 \to S_4$ be given. f is (computably) reducible to g with respect to representations $(\delta_1, \delta_2, \delta_3, \delta_4)$ (written "$f \leq_c g$ w.r.t. $(\delta_1, \delta_2, \delta_3, \delta_4)$") if there are a $(\delta_1, \delta_4, \delta_2)$-computable function $\mathbf{a} :\subseteq S_1 \times S_4 \to S_2$ and a (δ_1, δ_3)-computable function $\mathbf{b} :\subseteq S_1 \to S_3$ such that

$$f(x) = \mathbf{a}(x, g \circ \mathbf{b}(x))$$

for all $x \in \text{dom}(f)$.

For $S_1 = S_2 = \mathbb{N}^{\mathbb{N}}$ we speak simply of reducibility w.r.t. (δ_3, δ_4), because f can be identified with some of its $(\delta_{\mathbb{B}}, \delta_{\mathbb{B}})$-realizations. The same holds of $S_3 = S_4 = \mathbb{N}^{\mathbb{N}}$.

By [Bra,2005]:

$f \leq_c g$ w.r.t. $(\delta_1, \delta_2, \delta_3, \delta_4)$ $\wedge$ g is $\mathbf{\Sigma}^0_{k+1}$-computable w.r.t. (δ_3, δ_4) $\Rightarrow$ f is $\mathbf{\Sigma}^0_{k+1}$-computable w.r.t. (δ_1, δ_2).

Σ_k^0-completeness.

For any $k \in \mathbb{N}$ let $C_k : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ be the function:

$$C_k(p)(n) = \begin{cases} 0 & \text{if } \exists n_k \forall n_{k-1} \exists n_{k-2} ... Q n_1 : p\langle n, n_k, n_{k-1} ..., n_1\rangle \neq 0 \\ 1 & \text{otherwise} \end{cases}$$

where $Qn_1 = \exists n_1$ if k is odd, and $Qn_1 = \forall n_1$ else.

Given represented sets (S_i, δ_i), for $i = 0, 1$, a function $f :\subseteq S_1 \to S_0$ is $\mathbf{\Sigma}^0_{k+1}$-complete w.r.t. (δ_1, δ_0) if it is $\mathbf{\Sigma}^0_{k+1}$-computable w.r.t. (δ_1, δ_0) and $C_k \leq_c f$ w.r.t. (δ_1, δ_0).

By [Bra,2005]:

- given any function $g :\subseteq S_3 \to S_4$, one has that $g \leq_c C_k$ w.r.t. (δ_3, δ_4) if and only if g is $\mathbf{\Sigma}^0_{k+1}$-computable w.r.t. (δ_3, δ_4);

- for any $k \in \mathbb{N}$, the function C_k is $\mathbf{\Sigma}^0_{k+1}$-computable but not $\mathbf{\Sigma}^0_k$-computable.

Representation of continuous functions in Baire space.

Consider the set:

$$F^{\omega\omega} = \{F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}} : \ F \text{ is } (\mathbb{B}, \mathbb{B})\text{-continuous and } \mathrm{dom}(F) \text{ is a } G_\delta\text{-set}\}.$$

In the following, let $\eta :\subseteq \mathbb{N}^{\mathbb{N}} \to F^{\omega\omega}$ be any standard representation of $F^{\omega\omega}$ (see [Weih,2000]) satisfying

- the universal Turing machine (utm-)property;
- the parameters (smn-) property;
- any computable function $F \in F^{\omega\omega}$ has some computable η-name.

Urysohn Lemma is Σ^0_2-computable

Positive information for closed sets.

ψ_+ is the representation of the class $\mathcal{A}$ of the closed subsets of M defined in the following way: for $p \in \mathbb{N}^{\mathbb{N}}$ let

$$\psi_+(p) = A \in \mathcal{A} \Leftrightarrow \{n > 0 : n \in p\} = \{n : A \cap I_n^{\mathrm{M}} \neq \emptyset\}.$$

Let $\delta_{\mathrm{M}\mathbb{R}}$ be the representation of the set $C(M)$ of all total continuous real functions $f : M \to \mathbb{R}$ defined as follows for all $p \in \mathrm{dom}(\eta)$:

$$\delta_{\mathrm{M}\mathbb{R}}(p) = f \in C(M) \Leftrightarrow \eta_p \in F^{\omega\omega} \text{is a } (\delta_{\mathrm{M}}, \rho)\text{-realization of } f.$$

Lemma 1. There is a $\mathbf{\Sigma}^0_2$-computable function $o : \mathcal{A} \to C(M)$ w.r.t. $(\psi_+, \delta_{\mathrm{M}\mathbb{R}})$ mapping any closed set $A \subseteq M$ to some continuous function $o_{A} : M \to \mathbb{R}$ such that $A = o_{A}^{-1}[\{0\}]$.

Proof. Let for any not empty closed set A the function d_A be the distance function of A: $d_A(x) = \inf\{d(x,y) : y \in A\}$ for all $x \in M$.
Define now:

$$o_{A}(x) = \begin{cases} \min\{1, d_A(x)\} & \text{if } A \neq \emptyset \\ 1 & \text{otherwise} \end{cases}$$

Let p_A be a p_+-name of A and r_{x} be a δ_{M}-name of $x \in M$.
There is a $\mathbf{\Sigma}^0_2$-computable function L mapping p_{A} to some list of all ν_{M}-names of all basic open balls not intersecting A.
There is a computable function $\xi(p_1, p_2, p_3)$ which computes a ρ-name of $o_{A}(x)$ given as input p_{A}, $L(p_{A})$ and r_{x}.
By smn-property, there is a computable function S such that

$$\eta_{S(p_A, L(p_A))}(r_{x}) = \xi(p_{A}, L(p_{A}), r_{x}).$$

Hence $S(p_{A}, L(p_{A}))$ is a $\delta_{\mathrm{M}\mathbb{R}}$-name of o_{A} and the mapping $\lambda p_{A}.S(p_{A}, L(p_{A}))$ is $\mathbf{\Sigma}^0_2$-computable.

Theorem 1. There is a $\mathbf{\Sigma}^0_2$-computable function $u :\subseteq \mathcal{A} \times \mathcal{A} \to C(M)$ w.r.t. $(\psi_+, \psi_+, \delta_{\mathrm{M}\mathbb{R}})$, mapping every disjoint pair of closed subsets of M to some total continuous function $u_{A,B} : M \to \mathbb{R}$ such that $u_{A,B}(x) = 0$ for $x \in A$, $u_{A,B}(x) = 1$ for $x \in B$, and $0 < u_{A,B}(x) < 1$ otherwise.

Proof. For any closed set A let o_A be defined as in the proof of Lemma 1. Consider the function $u_{A,B}$ for $A, B \in \mathcal{A}$:

$$u_{A,B} = \frac{o_A}{o_A + o_B}.$$

Lemma 1 together with [Bra,2005] give then the result for $u : (A, B) \mapsto u_{A,B}$.

Proposition 1. For $\mathrm{M} = \mathbb{R}$, it obtains $C_1 \leq_c u$ w.r.t. $(\psi_+, \psi_+, \delta_{\mathbb{R}\mathbb{R}})$.

Proof. Let $p \in \mathbb{N}^{\mathbb{N}}$ be given. For any $n \in \mathbb{N}$ define the closed sets $A_n, B_n \subseteq \mathbb{R}$:

$$\begin{aligned} A_n &= \overline{\{n + 2^{-(k+2)} : \exists m \leq k \, (p\langle n, m\rangle \neq 0)\}}, \\ B_n &= \overline{\{n - 2^{-(k+2)} : \forall m \leq k \, (p\langle n, m\rangle = 0)\}}. \end{aligned}$$

Then put $A = \bigcup_{n\in\mathbb{N}} A_n$ and $B = \bigcup_{n\in\mathbb{N}} B_n$. Let G, G' be two computable functions such that $\psi_+(G(p)) = A$, $\psi_+(G'(p)) = B$. Let $u_{A,B} \in C(M)$ be such that $u_{A,B}[A] = \{0\}$, $u_{A,B}[B] = \{1\}$. Given $n \in \mathbb{N}$, if there is an m for which $p\langle n, m\rangle \neq 0$ then $n \in A$, whence $u_{A,B}(n) = 0$. But if such an m does not exist, then $n \in B$ and $u_{A,B}(n) = 1$.

Dieudonné's function for Urysohn-Tietze Lemma is Σ^0_2-computable

In [Weih,2001], Weihrauch observes that Dieudonné's approach to Urysohn-Tietze Lemma is not computable with respect to negative information.

We analyze again Dieudonné's solution, but for the case of positive information, and by the tools of effective Borel measurability.

Positive information for continuous partial functions.

Let $C^p(M)$ be the set of all continuous partial real functions $f :\subseteq M \to \mathbb{R}$ with closed domain and let $\delta^p_{\mathrm{M}\mathbb{R}}$ be the representation of such set defined in the following way:

$$\delta^p_{\mathrm{M}\mathbb{R}}\langle p, q\rangle = f \Leftrightarrow \eta_p \text{ is a } (\delta_{\mathrm{M}}, \rho)\text{-realization of } f \text{ and } \mathrm{dom}(f) = \psi_+(q).$$

Lemma 2 Let $f : M \times A \to \mathbb{R}$ be some continuous bounded real function, with $A \subseteq M$ closed, and let $h = \lambda x. \inf\{f(x,y) : y \in A\}$ is continuous. Then there is a function mapping f to h which is $\mathbf{\Sigma}^0_2$-computable w.r.t. $\left(\delta^p_{\mathrm{MM}\mathbb{R}}, \delta_{\mathrm{M}\mathbb{R}}\right)$.

Proof. Let $x \in M$, $y \in A$, respectively. Using utm-property is possible to compute a ρ-name of $f(x,y)$.
Obviously, $\alpha \in \mathbb{Q}$ is bigger than $h(x)$ if and only if there exists some $y \in A$ such that $f(x,y) < \alpha$.
Given then a list s of all rational numbers bigger than $h(x)$, a list t of all rational numbers smaller than $h(x)$ is easily obtained using $\lim_{\mathbb{B}}$. Since the set A may be uncountable and the function $\lim_{\mathbb{B}}$ is $\mathbf{\Sigma}^0_2$-computable, t may not be computable on the given input. Therefore the smn-property is bound to depend also on such argument. But the information coded in t depends on x, whereas by applying the smn-property we want to find a $\delta_{\mathrm{M}\mathbb{R}}$-name of h which depends only on $\langle p, q_A \rangle$. Hence, what we actually do is that we compute an "oracle" $K\langle p, q_A \rangle$ for the function h which is defined on suitable initial segments of δ_{M}-names, and such that $\xi(K\langle p, q_A \rangle, r_x)$ is a ρ-name of $h(x)$, for some computable function $\xi(p_1, p_2)$.

Theorem 2 There is a $\boldsymbol{\Sigma}^0_2$-computable function t w.r.t. $\left(\delta^p_{\mathrm{M}\mathbb{R}}, \delta_{\mathrm{M}\mathbb{R}}\right)$ mapping each partial continuous real function $f :\subseteq M \to [1,2]$ with closed domain and $\min(f) = 1, \max(f) = 2$ to some continuous total extension $g : M \to [1,2]$.

Proof. Let $\mathrm{dom}(f) = A$. Consider the Dieudonné function $f \mapsto g$, where g is defined by:

$$g(x) = \begin{cases} f(x) & \text{if } x \in A \\ \dfrac{\inf_{y \in A}\{f(y)d(x,y)\}}{d_A(x)} & \text{otherwise.} \end{cases}$$

The function g is a total extension of f and $\min(f) = \min(g) = 1$, $\max(f) = \max(g) = 2$. Moreover, g is continuous, as proven by Dieudonné in [Dieud,1960]. Let $\langle p, q_A \rangle$ be a $\delta^p_{\mathrm{M}\mathbb{R}}$-name of f. Let H be a $\boldsymbol{\Sigma}^0_2$-computable function such that $H\langle p, q_A \rangle$ is a $\delta_{\mathrm{M}\mathbb{R}}$-name of the function $\lambda x. \inf_{y \in A}\{f(y)d(x,y)\}$, for $x \in M$. $d_A(x)$ is computable given q_A and some list (of all the ν_{M}-names) of the denoted open balls not intersecting A. There is a suitable $\boldsymbol{\Sigma}^0_2$-computable function L which provides such a list given q_A. The map $\lambda\langle p, q_A \rangle.(H\langle p, q_A \rangle, L(q_A))$ is therefore $\boldsymbol{\Sigma}^0_2$-computable.

We then define a Turing machine $\mathcal{M}(p_1, p_2, p_3, p_4)$, which, on the input

$$(\langle p, q_{_A}\rangle, H\langle p, q_{_A}\rangle, L(q_{_A}), r_{_x}),$$

computes a ρ-name of $g(x)$, with $r_{_x}$ a $\delta_{_\mathrm{M}}$-name of $x \in M$. To define $\mathcal{M}$ we partly modify the original proof of the continuity of g given by Dieudonné . Let

$$i(x) = \frac{\inf_{y \in A}\{f(y)d(x,y)\}}{d_{_A}(x)}.$$

The intuitive idea is to construct a machine $\mathcal{M}$ which applies always f to x, unless it realizes at some stage that $x \notin A$. If so, the computation goes on applying i to x. The problem is to handle the process carefully, so that if we realize at a certain stage that the wrong function (i.e. f) has been applied to x, we are still in time to compute a name of $g(x) = i(x)$. This means that despite of having applied the wrong function, we have listed on the output tape only names of balls containing $i(x)$. If we succeed, we do not need to know at the beginning of the computation whether $x \in A$ or not, in order to make a choice between f and i. Such a knowledge may not be computably achievable with the information coded in the input and it is (partially) dependent on x. On the contrary, we want to find, by the smn-property, a possible name for a realization of the Dieudonné function, and this must be independent from x.

Let H be any function such that $H\langle p, q_A\rangle$ is a $\delta_{\mathrm{M}\mathbb{R}}$-name of the function $\lambda x.\,\mathrm{inf}_{y\in A}\{f(y)d(x,y)\}$, for $x \in M$.

We know there is a suitable $\mathbf{\Sigma}^0_2$-computable function L mapping any p_+-name of A to some $\delta_{\mathrm{M}\mathbb{R}}$-name of d_A.

The machine $\mathcal{M}$ is defined by induction on the number of stages, by formulating an effective version of the original Dieudonné's proof of continuity of g.

Finally let $\xi(p_1, p_2, p_3, p_4)$ be the function computed by $\mathcal{M}$. By the smn-property there is a computable function S such that

$$\eta_{S(\langle p,q_A\rangle, H\langle p,q_A\rangle, L(q_A))}(r_x) = \xi(\langle p, q_A\rangle, H\langle p, q_A\rangle, L(q_A), r_x).$$

Then the function $\langle p, q_A\rangle \mapsto S(\langle p, q_A\rangle, H\langle p, q_A\rangle, L(q_A))$ is $\mathbf{\Sigma}^0_2$-computable.

Proposition 2. The Dieudonné function t is not computable: in some cases it is $\mathbf{\Sigma}^0_2$-complete (w.r.t . $\left(\delta^p_{\mathbf{M}\mathbb{R}}, \delta_{\mathbf{M}\mathbb{R}}\right)$).

Proof. Consider the computable function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that $f(x, y) = |x - 1| + 1$ for all $x, y \in \mathbb{R}$. This function f has a computable $\delta_{\mathbb{R}\mathbb{R}\mathbb{R}}$-name, say $r \in \mathbb{N}^{\mathbb{N}}$. Let $x_n = (0; n)$, $y_n = (1; n)$, $z_n = (2; n)$ for all $n \in \mathbb{N}$ and take a computable function $H : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that for any $p \in \mathbb{N}^{\mathbb{N}}$:

$$\psi_+(H(p)) = \{x_n, y_n : n \in \mathbb{N}\} \cup \{z_n : \exists m(p\langle n, m\rangle \neq 0)\}.$$

Put $\psi_+(H(p)) = A$. Then $\langle r, H(p)\rangle$ is a $\delta^p_{\mathbb{R}\mathbb{R}\mathbb{R}}$-name of $f_{|A}$.
Consider the Dieudonné extension g of $f_{|A}$. For any $n \in \mathbb{N}$, if there is an m such that $p\langle n, m\rangle \neq 0$ then $g(z_n) = f_{|A}(z_n) = f(z_n) = 2$. If there is no such m then $g(z_n) = f_{|A}(y_n)d(y_n, z_n) = f(y_n)d(y_n, z_n) = 1$.

Referencies

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Appendix

The construction of $\mathcal{M}$ in Theorem 2.

We define $\mathcal{M}$ by induction on the number of stages.

- Stage 0) The machine outputs nothing.

- Stage s) Suppose $\mathcal{M}$ has listed in the output tape only names of balls containing $g(x)$. Now $\mathcal{M}$ must add on the output tape the name of a denoted open ball $B^{\mathbb{R}} \subseteq \mathbb{R}$ containing $g(x)$ and with a diameter smaller than or equal to 2^{-s}. Let u be the initial segment of $r_{_x}$ that has been analyzed by $\mathcal{M}$ until now. Suppose every ball listed in u intersects A. Therefore we are sure that u is an initial segment of a point in A. Then $\mathcal{M}$ applies, by utm-property, η_p to $r_{_x}$ until it finds by $r_{_x}$ some open ball $I_n^{\mathbb{M}}$ (intersecting A) mapped by f inside some ball $I_m^{\mathbb{R}} \subseteq \mathbb{R}$ with a diameter smaller than $2^{-(s+2)}$.

Suppose such a ball I_n^{M} exists and let $I_n^{\mathrm{M}} = B(c, \alpha)$ for $c \in Q, \alpha \in \mathbb{Q}^+$. Suppose $\mathcal{M}$ finds also another denoted open ball $I_k^{\mathrm{M}} = B(e, \beta)$ such that $x \in I_k^{\mathrm{M}} \subseteq I_n^{\mathrm{M}}$, $I_k^{\mathrm{M}} \cap A \neq \emptyset$, and

$$\alpha - d(c, e) - \beta > 4\beta.$$

If $x \in A$ then both I_n^{M} and I_k^{M} exist. Now we see how to find a suitable ball $B^{\mathbb{R}}$ containing $g(x)$ (independently on whether $x \in A$ or not). Let $C = A \cap I_n^{\mathrm{M}}$ and $D = A - C$. Since $x \in I_k^{\mathrm{M}} \subseteq I_n^{\mathrm{M}}$, $I_k^{\mathrm{M}} \cap A \neq \emptyset$, $diam(I_k^{\mathrm{M}}) = 2\beta$, there is some $y \in C$ such that $d(x, y) < 2\beta$. But for any $y \in D$:

$$d(x, y) \geq d(y, c) - d(c, e) - d(e, x) > \alpha - d(c, e) - \beta > 4\beta.$$

Therefore

$$d_A(x) = d_C(x) = \inf_{y \in C}\{d(x, y)\} \tag{1}$$

Moreover, for $y \in C \cap I_k^{\mathrm{M}}$: $f(y)d(x, y) < 4\beta$, while for $y \in D$: $f(y)d(x, y) > 4\beta$. So

$$\inf_{y \in C}\{f(y)d(x, y)\} = \inf_{y \in A}\{f(y)d(x, y)\}. \tag{2}$$

Recall that for any $y, z \in C$: $|f(y) - f(z)| < 2^{-(s+2)}$, thus $f(z) - 2^{-(s+2)} < f(y) < f(z) + 2^{-(s+2)}$. Therefore, chosen $z \in C$, for any $y \in C$:

$$(f(z) - 2^{-(s+2)})d(x,y) \leq f(y)d(x,y) \leq (f(z) + 2^{-(s+2)})d(x,y),$$

hence

$$(f(z) - 2^{-(s+2)}) \inf_{y \in C}\{d(x,y)\} = \inf_{y \in C}\{(f(z) - 2^{-(s+2)})d(x,y)\} \leq \inf_{y \in C}\{f(y)d(x,y)\}$$

and

$$\inf_{y \in C}\{f(y)d(x,y)\} \leq \inf_{y \in C}\{(f(z) + 2^{-(s+2)})d(x,y)\} = (f(z) + 2^{-(s+2)}) \inf_{y \in C}\{d(x,y)\}.$$

By (1) $d_A(x) = \inf_{y \in C}\{d(x,y)\}$, and so by (2) we conclude:

$$(f(z) - 2^{-(s+2)})d_A(x) \leq \inf_{y \in A}\{f(y)d(x,y)\} \leq (f(z) + 2^{-(s+2)})d_A(x)$$

which proves that $|g(x) - f(z)| \leq 2^{-(s+2)}$. Indeed if $g(x) = f(x)$ then $|g(x) - f(z)| < 2^{-(s+2)}$ by our hypothesis that $f[I_n^{\mathbb{M}}] \subseteq I_m^{\mathbb{R}}$ and $diam(I_m^{\mathbb{R}}) < 2^{-(s+2)}$.

Else:

$$f(z) - 2^{-(s+2)} \leq \frac{\inf_{y \in A}\{f(y)d(x,y)\}}{d_A(x)} = g(x) \leq f(z) + 2^{-(s+2)}.$$

Let $\gamma \in \mathbb{Q}$ be the center of $I_m^{\mathbb{R}}$. Then $|\gamma - g(x)| \leq |\gamma - f(z)| + |f(z) - g(x)| < 2^{-(s+1)}$. The machine $\mathcal{M}$ can let so $B^{\mathbb{R}} = B(\gamma, 2^{-(s+1)})$.
Suppose otherwise that either I_n^{M} or I_k^{M} is not defined. Then $x \notin A$ and $\mathcal{M}$ recognizes this, sooner or later, through $L(q_A)$. By induction hypothesis, any ball named in the output tape at stage $s-1$ contains $g(x)$. Using computability of $\div$, the value $i(x)$ is computable via utm-property applied to $(H\langle p, q_A\rangle, r_x)$ and $(q_A, L(q_A), r_x)$. So now $\mathcal{M}$ computes $i(x)$ until it finds some ball with a diameter smaller than or equal to 2^{-s} and writes its name on the output tape.
$\mathcal{M}$ proceeds to compute $i(x)$ similarly at any other stage $u > s$.