Brownian motion and Kolmogorov complexity

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1. Introduction

The results of this talk follows on my papers [F,JSL] and [F,AM], where it was shown that each binary string α which is complex (random) in the sense of Kolmogorov-Chaitin (a KC-string) can be algorithmically transformed into a "generic" Brownian motion x_α.
It is generic in the sense that every probabilistic event which holds almost surely with respect to the Wiener measure, is reflected in x_α, provided the probabilistic event has a suitably effective description.

2. Complex oscillations

The class of generic Brownian motions coincides with a class $\mathcal{C}$ of functions which were introduced by Asarin and Prokovskiy in 1986.
Each function in the latter class is a uniform limit of a sequence (x_n) of piecewise linear functions.
Moreover, every x_n can be encoded by a binary string s_n of length n such that, for some positive constant d, the Kolmogorov complexity of s_n is at least $n - d$, for all large values of n.

For this reason we called the elements of $\mathcal{C}$ *complex oscillations*. The complex oscillations have interesting recursion-theoretic properties. For example, it is shown in [F,JSL] that, if x is a complex oscillsation and r is a nonzero recursive real number in the unit interval, then $x(r)$ will not be a real number.
In [F, AM] I showed that, for each $x \in \mathcal{C}$, one can compute from the values of x at the rational numbers a unique KC-string α such that $x = x_\alpha$.

3. Homogeneous structures given by generic Brownian motion

In this way one can identify interesting implicit structure in a generic Brownian motion.
For example, the codes of many countable homogeneous relational structures can be computed from the values of a generic Brownian motion at the rationals in the unit interval.
Recall that a relational structure X is homogeneous if any isomorphism $f : A \to B$ between finite substructures of X can be extended to an automorphism of X.

The universal procedure which computes from the values of a complex oscillation x the KC-string α such that $x = x_\alpha$, also yields a code of a very interesting homogeneous structure, the so-called Rado graph or random graph.
Indeed, if α is a KC-string and $e_1, e_2, \ldots$ is a recursive enumeration, without repetition, of the 2-element subsets of ω, let $R_\alpha = (\omega, E_\alpha)$ be the graph defined by:

$$e_i \in E_\alpha \leftrightarrow \alpha_i = 1.$$

Then the graph R_α is isomorphic to Rado's graph [F, 1996].
In this sense one could say that a Rado graph is "enfolded" in every complex oscillation.

In the paper on which this talk is based, we take a closer look at the reverse process, namely the unfolding of KC-strings, not only to a generic Brownian motion as in [F,AM], but also to the dynamical aspects of Brownian motion, as reflected in every complex oscillation. Our focus is on the structure of the so-called rapid points of a complex oscillation. The fractal geometry of a

4. Rapid points

Call a point $t \in (0,1)$ a *rapid point* of a continuous function X on the unit interval when

$$\overline{\lim_{h \to 0}} \frac{|X(t+h) - X(t)|}{\sqrt{|h| \log(1/|h|)}} > 0.$$

Denote the set of rapid points of X by $R(X)$. It was shown by Orey and Taylor (1974) that Brownian motion has almost surely a set of rapid points of Hausdorff dimension 1.

4. Fractal properties of rapid points

When X is one-dimensional Brownian motion, the set $R(X)$ has an extremely interesting structure. For example, Kaufmann showed in 1974 that, almost surely, $R(X)$ contains, for each $0 < \beta < 1$, a Salem set of Hausdorff dimension β.
(Recall that a compact subset E of $\mathbf{R}^d$ of Hausdorff dimension $\beta > 0$ is said to be a Salem set, if β is the supremum of the reals $0 \leq \alpha < d$ for which there is some positive nonzero Radon measure μ with support contained in E, such that the Fourier transform $\hat{\mu}$ of μ satisfies $|\hat{\mu}(\xi)|^2 \ll |\xi|^{-\alpha}$, for all large values of $|\xi|$. In this case, E will generate $\mathbf{R}^d$ as an abelian group!)

6. Recursive properties of rapid points

The rapid points of a complex oscillation have a specific recursive structure. If x is a complex oscillation, then x has a dense set of rapid points.

If x is the complex oscillation x_α associated with the KC-string α, a dense set of rapid points can be effectively retrieved from α. *Indeed, there is a universal algorithmic procedure which, upon having access to an oracle for a KC-string α, will yield, for any closed dyadic interval I, a sequence (t_k) of rational numbers in I such that $|t_{k+1} - t_k| < 2^{-k}$ for all $k \geq 1$ and ,moreover, such that the limit t of the sequence (t_k) is a rapid point of the complex oscillation x_α associated with α.*

Furthermore, each rapid point of a complex oscillation is *not* a recursive real number.
In fact, if $t \in (0, 1)$ is a recursive real number, then t is an "ordinary" point of x. This means that Khintchine's law of the iterated logarithm is reflected in x at every recursive t, i.e., if t is recursive, then

$$\overline{\lim_{h \to 0}} \frac{|x(t+h) - x(t)|}{\sqrt{2|h| \log \log(1/|h|)}} = 1.$$

Briefly the construction of x_α from a KC-string α is as follows: Let $g : (0,1) \to \mathbf{R}$ be the function defined by

$$\alpha = \int_{-\infty}^{g(\alpha)} \frac{e^{-t^2/2}}{\sqrt{2\pi}} dt, \quad \alpha \in (0,1).$$

Note that g is a recursive function, i.e., there is a uniform procedure that outputs $g(\alpha)$ up to arbitrary accuracy using only a finite number of bits of α. We fix a recursive bijection $<,>$ from ω^2 to ω. To any $\alpha \in \mathcal{N}$, we associate a sequence $B = (\beta_0, \beta_1, \beta_{jn} : j \geq 1, 0 \leq n < 2^j)$, where the sequence (β_{jn}) is lexicographically ordered with respect to the double indices jn, in such a way that the kth term of the sequence B is given by

$$\alpha_{k0}\alpha_{k1}\cdots .$$

Here, we have written kl instead of $< k, l >$. For $1 \leq j < \omega$, $0 \leq n < 2^j$, set $\xi_{jn} = g(\beta_{jn})$; in addition, set $\xi_k = g(\beta_k)$, for $k = 0, 1$. It follows that there is a uniform procedure that computes from $\alpha \in KC$, for each j, n, the number ξ_{jn} up to arbitrary accuracy. For $\alpha \in \mathcal{N}$ and $t \in [0, 1]$ set

$$x_\alpha(t) = \xi_0 \Delta_0(t) + \xi_1 \Delta_1(t) + \sum_{j<\omega} \sum_{n<2^j} \xi_{jn} \Delta_{jn}(t).$$

It is shown in [F, AM] that, if $\alpha \in KC$, then the series converges and that the function x_α is in fact a complex oscillation. Conversely, for every complex oscillation x, there is a unique KC-string α such that $x = x_\alpha$.

Theorem

If (A_k) is a uniform sequence of $\Sigma^0_1(\mathcal{F})$ sets with $\sum_k W(A_k) < \infty$, then, for each complex oscillation x, it is the case that $x \notin A_k$ for all large values of k.

An analogue for KC-strings of this theorem appears in [3].

- Asarin, E.A. and Prokovskiy, A.V.: Primeenenie kolmogorovskoi slozhnosti k anlizu dinamiki upravlemykh sistem, *Automatika i Telemekhanika* **1** (1986), 25-33.
- Chaitin, G.J.: On the length of programs for computing binary sequences, *J. Assoc. Comput. Mach.* **13** (1966), 547-569.
- Chaitin, G.A.: *Algorithmic information theory*, Cambridge University Press, 1987.
- Fouché, W.L.: Descriptive complexity and reflective properties of combinatorial configurations, *J. Lond. Math. Soc.* **54** (1996), 199-208.
- Fouché, W.L.: Arithmetical representations of Brownian motion I, *J. Symb. Logic* **65** (2000), 421-442.

- Fouché, W.L. : The descriptive complexity of Brownian motion, *Advances in Mathematics* **155** (2000), 3177-343.
- Fouché, W.L. : Fractal geometry of Brownian motions of high Kolmogorov complexity, in preparation.
- Freedman, D.: *Brownian motion and diffusion*, Holden-Day, 1971.
- Hida, T.: *Brownian motion*, Springer-Verlag, New York, 1980.
- Hinman, P.G.: *Recursion-theoretic hierarchies*, Springer-Verlag, New York,1978.
- Kaufman, R.: Large increments of Brownian motion, *Nagoya Math. J.* **56** (1974), 139-145.

- Kahane, J.-P.: Sur l'irrégularité locale du mouvement brownien, *C. r. hebd. séanc. Acad. Sci., Paris* **278** (1974), 331-333.
- Kahane, J.-P.: *Some random series of functions* (second edition), Cambridge University Press, 1993.
- Khinchine, A.Y.: *Asymptotische Gesetze der Wahrscheinlichkeitsrechnung*. Ergebn. Math. **2**, Berlin, 1933.
- Kolmogorov, A.N.: Three approaches to the quantitative definition of randomness, *Probl. Inform. Transmission* **1** (1965), 1-7.
- Martin-Löf, P.: The definition of random sequences, *Information and Control* **9** (1966), 602-619.

- Orey, S. and Taylor, S.J.: How often on a Brownian path does the law of iterated logarithm fail?, *Proc. Lond. Math. Soc* **28** (1974), 174-192.
- Rado, R.: Universal graphs and universal functions, *Acta Arith.* **9** (1964), 393-407.
- Shen, A.Kh.: Connections between different algorithmic definitions of randomness, *Soviet Math. Dokl.* **38** (1989), 316-319.
- van Lambalgen, M.: Von Mises' definition of random sequences reconsidered, *J. Symb. Logic* **52** (1987), 725-755.
- Vitányi, P. and Li, M.: *An introduction to Kolmogorov complexity and its applications*, Springer-Verlag, New York, 1993.
- Vov'k, V.G.: The law of the iterated logarithm for