Brownian motion and Kolmogorov complexity

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The results of this talk follows on my papers [F,JSL] and [F,AM], where it was shown that each binary string $\alpha$ which is complex (random) in the sense of Kolmogorov-Chaitin (a $KC$-string) can be algorithmically transformed into a “generic” Brownian motion $x_\alpha$.

It is generic in the sense that every probabilistic event which holds almost surely with respect to the Wiener measure, is reflected in $x_\alpha$, provided the probabilistic event has a suitably effective description.
The class of generic Brownian motions coincides with a class $C$ of functions which were introduced by Asarin and Prokovskiy in 1986. Each function in the latter class is a uniform limit of a sequence $(x_n)$ of piecewise linear functions. Moreover, every $x_n$ can be encoded by a binary string $s_n$ of length $n$ such that, for some positive constant $d$, the Kolmogorov complexity of $s_n$ is at least $n - d$, for all large values of $n$. 
For this reason we called the elements of $\mathcal{C}$ *complex oscillations*. The complex oscillations have interesting recursion-theoretic properties. For example, it is shown in [F,JSL] that, if $x$ is a complex oscillation and $r$ is a nonzero recursive real number in the unit interval, then $x(r)$ will not be a real number.

In [F, AM] I showed that, for each $x \in \mathcal{C}$, one can compute from the values of $x$ at the rational numbers a unique $KC$-string $\alpha$ such that $x = x_{\alpha}$. 
3. Homogeneous structures given by generic Brownian motion

In this way one can identify interesting implicit structure in a generic Brownian motion. For example, the codes of many countable homogeneous relational structures can be computed from the values of a generic Brownian motion at the rationals in the unit interval. Recall that a relational structure \( X \) is homogeneous if any isomorphism \( f : A \to B \) between finite substructures of \( X \) can be extended to an automorphism of \( X \).
The universal procedure which computes from the values of a complex oscillation $x$ the $KC$-string $\alpha$ such that $x = x_\alpha$, also yields a code of a very interesting homogeneous structure, the so-called Rado graph or random graph.

Indeed, if $\alpha$ is a $KC$-string and $e_1, e_2, \ldots$ is a recursive enumeration, without repetition, of the 2-element subsets of $\omega$, let $R_\alpha = (\omega, E_\alpha)$ be the graph defined by:

$$e_i \in E_\alpha \leftrightarrow \alpha_i = 1.$$ 

Then the graph $R_\alpha$ is isomorphic to Rado’s graph [F, 1996]. In this sense one could say that a Rado graph is “enfolded” in every complex oscillation.
In the paper on which this talk is based, we take a closer look at the reverse process, namely the unfolding of $KC$-strings, not only to a generic Brownian motion as in [F,AM], but also to the dynamical aspects of Brownian motion, as reflected in every complex oscillation. Our focus is on the structure of the so-called rapid points of a complex oscillation. The fractal geometry of a...
4. Rapid points

Call a point \( t \in (0,1) \) a \textit{rapid point} of a continuous function \( X \) on the unit interval when

\[
\lim_{h \to 0} \frac{|X(t + h) - X(t)|}{\sqrt{|h| \log(1/|h|)}} > 0.
\]

Denote the set of rapid points of \( X \) by \( R(X) \). It was shown by Orey and Taylor (1974) that Brownian motion has almost surely a set of rapid points of Hausdorff dimension 1.
When $X$ is one-dimensional Brownian motion, the set $R(X)$ has an extremely interesting structure. For example, Kaufmann showed in 1974 that, almost surely, $R(X)$ contains, for each $0 < \beta < 1$, a Salem set of Hausdorff dimension $\beta$. (Recall that a compact subset $E$ of $\mathbb{R}^d$ of Hausdorff dimension $\beta > 0$ is said to be a Salem set, if $\beta$ is the supremum of the reals $0 \leq \alpha < d$ for which there is some positive nonzero Radon measure $\mu$ with support contained in $E$, such that the Fourier transform $\hat{\mu}$ of $\mu$ satisfies $|\hat{\mu}(\xi)|^2 \ll |\xi|^{-\alpha}$, for all large values of $|\xi|$. In this case, $E$ will generate $\mathbb{R}^d$ as an abelian group!)
The rapid points of a complex oscillation have a specific recursive structure. If $x$ is a complex oscillation, then $x$ has a dense set of rapid points.

If $x$ is the complex oscillation $x_\alpha$ associated with the $KC$-string $\alpha$, a dense set of rapid points can be effectively retrieved from $\alpha$. Indeed, there is a universal algorithmic procedure which, upon having access to an oracle for a $KC$-string $\alpha$, will yield, for any closed dyadic interval $I$, a sequence $(t_k)$ of rational numbers in $I$ such that $|t_{k+1} - t_k| < 2^{-k}$ for all $k \geq 1$ and, moreover, such that the limit $t$ of the sequence $(t_k)$ is a rapid point of the complex oscillation $x_\alpha$ associated with $\alpha$. 
Furthermore, each rapid point of a complex oscillation is *not* a recursive real number. In fact, if $t \in (0, 1)$ is a recursive real number, then $t$ is an “ordinary” point of $x$. This means that Khintchine’s law of the iterated logarithm is reflected in $x$ at every recursive $t$, i.e., if $t$ is recursive, then

$$
\lim_{h \to 0} \frac{|x(t + h) - x(t)|}{\sqrt{2|h| \log \log(1/|h|)}} = 1.
$$
Briefly the construction of $x_\alpha$ from a $KC$-string $\alpha$ is as follows: Let $g : (0, 1) \rightarrow \mathbb{R}$ be the function defined by

$$\alpha = \int_{-\infty}^{g(\alpha)} \frac{e^{-t^2/2}}{\sqrt{2\pi}} dt, \quad \alpha \in (0, 1).$$
Note that $g$ is a recursive function, i.e., there is a uniform procedure that outputs $g(\alpha)$ up to arbitrary accuracy using only a finite number of bits of $\alpha$. We fix a recursive bijection $<,>$ from $\omega^2$ to $\omega$. To any $\alpha \in \mathbb{N}$, we associate a sequence $B = (\beta_0, \beta_1, \beta_{jn} : j \geq 1, 0 \leq n < 2^j)$, where the sequence $(\beta_{jn})$ is lexicographically ordered with respect to the double indices $jn$, in such a way that the $k$th term of the sequence $B$ is given by

$$\alpha_k0\alpha_k1 \cdots$$
Here, we have written $kl$ instead of $< k, l >$. For $1 \leq j < \omega$, $0 \leq n < 2^j$, set $\xi_{jn} = g(\beta_{jn})$; in addition, set $\xi_k = g(\beta_k)$, for $k = 0, 1$. It follows that there is a uniform procedure that computes from $\alpha \in KC$, for each $j, n$, the number $\xi_{jn}$ up to arbitrary accuracy. For $\alpha \in \mathcal{N}$ and $t \in [0, 1]$ set

$$x_{\alpha}(t) = \xi_0 \Delta_0(t) + \xi_1 \Delta_1(t) + \sum_{j<\omega} \sum_{n<2^j} \xi_{jn} \Delta_{jn}(t).$$

It is shown in [F, AM] that, if $\alpha \in KC$, then the series converges and that the function $x_{\alpha}$ is in fact a complex oscillation. Conversely, for every complex oscillation $x$, there is a unique $KC$-string $\alpha$ such that $x = x_{\alpha}$. 
Theorem

If \((A_k)\) is a uniform sequence of \(\Sigma_1^0(F)\) sets with \(\sum_k W(A_k) < \infty\), then, for each complex oscillation \(x\), it is the case that \(x \notin A_k\) for all large values of \(k\).

An analogue for \(KC\)-strings of this theorem appears in [3].


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Vov’k, V.G.: The law of the iterated logarithm for Brownian motion and Kolmogorov complexity.