

Relativized Degree Spectra

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Outline

- Enumeration of a structure
- Degree spectra and co-spectra
- Relatively α -intrinsic sets
- Relative spectra of structures
- Normal Form Theorem
- The connection with the Joint Spectra
- The Minimal pair theorem
- Quasi-minimal degrees

Enumeration of a structure

Let $\mathfrak{A} = (\mathbb{N}; R_1, \dots, R_k, =, \neq)$ be a countable abstract structure.

- An enumeration f of \mathfrak{A} is a total mapping from \mathbb{N} onto \mathbb{N} .
- for any $A \subseteq \mathbb{N}^a$ let
$$f^{-1}(A) = \{\langle x_1 \dots x_a \rangle : (f(x_1), \dots, f(x_a)) \in A\}.$$
- $f^{-1}(\mathfrak{A}) = f^{-1}(R_1) \oplus \dots \oplus f^{-1}(R_k) \oplus f^{-1}(=) \oplus f^{-1}(\neq).$



Degree spectra of structures

Definition

- **The Degree spectrum of \mathfrak{A}** is the set

$$DS(\mathfrak{A}) = \{d_e(f^{-1}(\mathfrak{A})) \mid f \text{ is an enumeration of } \mathfrak{A}\}.$$

- L. Richter [1981] — degrees of structures.
- J. Knight [1986] — jump degrees of structures.
- Ash, Jockush, Downey, Slaman, Soskov.

Definition

- **The Co-spectrum of \mathfrak{A}** is the set

$$CS(\mathfrak{A}) = \{\mathbf{b} : (\forall \mathbf{a} \in DS(\mathfrak{A}))(\mathbf{b} \leq \mathbf{a})\}.$$



Examples

- 1981 (Richter) Let $\mathfrak{A} = (\mathbb{N}; <, =, \neq)$ be a linear ordering.
- $DS(\mathfrak{A})$ contains a minimal pair of degrees, $CS(\mathfrak{A}) = \{\mathbf{0}_e\}$.
 - If $DS(\mathfrak{A})$ has a least element \mathbf{a} , then $\mathbf{a} = \mathbf{0}_e$.
- 1998 (Slaman) $DS(\mathfrak{A}) = \{\mathbf{a} : \mathbf{a} \text{ is total and } \mathbf{0}_e < \mathbf{a}\}$,
 $CS(\mathfrak{A}) = \{\mathbf{0}_e\}$.
- $DS(\mathfrak{A})$ has not a least element.
- 1998 (Coles, Downey, Slaman) Every principle ideal of enumeration degrees is a $CS(\mathfrak{A})$ for some torsion free abelian group \mathfrak{A} .
- 2002 (Soskov) Every countable ideal is a $CS(\mathfrak{A})$ for some \mathfrak{A} .



Definition

Let $\mathcal{A} \subseteq \mathcal{D}_e$. Then \mathcal{A} is *upwards closed* if

$$\mathbf{a} \in \mathcal{A}, \mathbf{b} \text{ is total and } \mathbf{a} \leq \mathbf{b} \Rightarrow \mathbf{b} \in \mathcal{A}.$$

The Degree spectra are upwards closed.

- General properties of upwards closed sets of degrees.
- Specific properties:
 - the Minimal pair type theorem;
 - the existence of Quasi-minimal degree.



Relatively α -intrinsic sets

1989 (Ash, Knight, Manasse, Slaman, Chisholm).

- The set A is *relatively α -intrinsic on \mathfrak{A}* if for every enumeration f of \mathfrak{A} the set $f^{-1}(A) \leq_e f^{-1}(\mathfrak{A})^{(\alpha)}$, $\alpha < \omega_1^{CK}$.

2002 (Soskov, Baleva)

- Let $\{B_\alpha\}_{\alpha \leq \zeta}$ be a sequence of subset of \mathbb{N} and $\zeta < \omega_1^{CK}$.
- Add each set B_α to the structure \mathfrak{A} as a new predicate which is relatively α -intrinsic on \mathfrak{A} .
- Restrict the class of all enumerations of \mathfrak{A} to the class of those enumerations f of \mathfrak{A} for which $f^{-1}(B_\alpha) \leq_e f^{-1}(\mathfrak{A})^{(\alpha)}$.



Relative Spectra of Structures

Let $\mathfrak{A}_1, \dots, \mathfrak{A}_n$ be arbitrary abstract structures on \mathbb{N} , $k \leq n$.
An enumeration f of \mathfrak{A} is **k-acceptable** with respect to the structures $\mathfrak{A}_1, \dots, \mathfrak{A}_k$, if

$$f^{-1}(\mathfrak{A}_1) \leq_e (f^{-1}(\mathfrak{A}))', \dots, f^{-1}(\mathfrak{A}_k) \leq_e (f^{-1}(\mathfrak{A}))^{(k)}.$$

Denote by \mathcal{E}_k the class of all k -acceptable enumerations.

Definition

The Relative spectrum of the structure \mathfrak{A} with respect to $\mathfrak{A}_1, \dots, \mathfrak{A}_n$ is the set

$$\text{RS}(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n) = \{d_e(f^{-1}(\mathfrak{A})) \mid f \in \mathcal{E}_n\}$$



Lemma

If F is a total set, $f \in \mathcal{E}_n$ and $f^{-1}(\mathfrak{A}) \leq_e F$, then there exists an enumeration $g \in \mathcal{E}_n$, such that

- 1 $g^{-1}(\mathfrak{A}) \equiv_e F \oplus f^{-1}(\mathfrak{A}) \equiv_e F$;
- 2 $g^{-1}(B) \leq_e F \oplus f^{-1}(B)$, for every $B \subseteq \mathbb{N}$.

Corollary

The Relative spectrum $RS(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n)$ is upwards closed.



Let $k \leq n$. **The k th Jump Relative spectrum** of \mathfrak{A} with respect to $\mathfrak{A}_1, \dots, \mathfrak{A}_n$ is the set

$$\text{RS}_k(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n) = \{\mathbf{a}^{(k)} \mid \mathbf{a} \in \text{RS}(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n)\}.$$

Proposition

The k th Jump Relative spectrum $\text{RS}_k(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n)$ is upwards closed.



Relative Co-spectra of Structures

Definition

The Relative co-spectrum of \mathfrak{A} with respect to $\mathfrak{A}_1, \dots, \mathfrak{A}_n$, is the co-set of $RS(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n)$, i.e.

$$CRS(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n) = \{\mathbf{b} \mid (\forall \mathbf{a} \in RS(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n))(\mathbf{b} \leq \mathbf{a})\}.$$

Let $k \leq n$. **The Relative k th co-spectrum** of \mathfrak{A} with respect to $\mathfrak{A}_1, \dots, \mathfrak{A}_n$, is the co-set of $RS_k(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n)$, i.e.

$$CRS_k(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n) = \{\mathbf{b} \mid (\forall \mathbf{a} \in RS_k(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n))(\mathbf{b} \leq \mathbf{a})\}.$$



The jump set

The jump set \mathcal{P}_k^f of \mathfrak{A} with respect to $\mathfrak{A}_1, \dots, \mathfrak{A}_n$:

- 1 $\mathcal{P}_0^f = f^{-1}(\mathfrak{A})$.
- 2 $\mathcal{P}_{k+1}^f = (\mathcal{P}_k^f)' \oplus f^{-1}(\mathfrak{A}_k)$.

Theorem

For every $A \subseteq \mathbb{N}$ and $k \leq n$, the following are equivalent:

- 1 $d_e(A) \in \text{CRS}_k(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n)$.
- 2 $A \leq_e \mathcal{P}_k^f$, for every k -acceptable enumeration f of \mathfrak{A} with respect to $\mathfrak{A}_1, \dots, \mathfrak{A}_k$.



The Normal Form Theorem

The set A is *formally k -definable* on \mathfrak{A} with respect to $\mathfrak{A}_1, \dots, \mathfrak{A}_n$ if there exists a recursive sequence $\{\Phi^{\gamma(x)}(W_1, \dots, W_r)\}$ of Σ_k^+ formulae and elements t_1, \dots, t_r of \mathbb{N} such that:

$x \in A \iff (\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n) \models \Phi^{\gamma(x)}(W_1/t_1, \dots, W_r/t_r)$.

- $\Sigma_0^+ : (\exists \bar{Y})(\beta_1 \ \& \ \dots \ \& \ \beta_k)$;
- $\Sigma_{k+1}^+ : \text{r.e. disjunction of } (\exists \bar{Y})\Phi(\bar{X}, \bar{Y}),$
 $\Phi = (\phi_1 \ \& \ \dots \ \& \ \phi_l \ \& \ \beta)$.

Theorem

A set $A \in \text{CRS}_k(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n)$ if and only if A is formally k -definable on \mathfrak{A} with respect to $\mathfrak{A}_1, \dots, \mathfrak{A}_n$.



The connection with the Joint Spectra

Definition

The *Joint spectrum* of $\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n$ is the set

$$DS(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n) = \{\mathbf{a} : \mathbf{a} \in DS(\mathfrak{A}), \mathbf{a}' \in DS(\mathfrak{A}_1), \dots, \mathbf{a}^{(n)} \in DS(\mathfrak{A}_n)\}.$$

- 1 $CS(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n) = CRS(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n)$.
- 2 There are structures \mathfrak{A} and \mathfrak{A}_1 , for which $CS_1(\mathfrak{A}, \mathfrak{A}_1) \neq CRS_1(\mathfrak{A}, \mathfrak{A}_1)$.
- 3 The difference:
 - $A \leq_e \mathcal{P}(f^{-1}(\mathfrak{A}), f_1^{-1}(\mathfrak{A}_1), \dots, f_n^{-1}(\mathfrak{A}_n))$ for every enumerations f of \mathfrak{A} , f_1 of $\mathfrak{A}_1, \dots, f_n$ of \mathfrak{A}_n .
 - in the normal form $(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n)$ — as a many-sorted structure with separated sorts.



Minimal Pair Theorem

Theorem

For any structures $\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n$, there exist enumeration degrees \mathbf{f} and \mathbf{g} in $\text{RS}(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n)$, such that for any enumeration degree \mathbf{a} and each $k \leq n$:

$$\mathbf{a} \leq \mathbf{f}^{(k)} \ \& \ \mathbf{a} \leq \mathbf{g}^{(k)} \Rightarrow \mathbf{a} \in \text{CRS}_k(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n).$$



Quasi-Minimal Degree

Soskov An enumeration degree \mathbf{q}_0 is *quasi-minimal with respect to* $DS(\mathfrak{A})$ if

- $\mathbf{q}_0 \notin CS(\mathfrak{A})$;
- for any total enumeration degree \mathbf{a} : $\mathbf{a} \geq \mathbf{q}_0 \Rightarrow \mathbf{a} \in DS(\mathfrak{A})$ and $\mathbf{a} \leq \mathbf{q}_0 \Rightarrow \mathbf{a} \in CS(\mathfrak{A})$.

Theorem

For any structures $\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n$ there exists an enumeration degree \mathbf{q} such that:

- 1 $\mathbf{q} \notin CRS(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n)$;
- 2 If \mathbf{a} is a total degree and $\mathbf{a} \geq \mathbf{q}$, then $\mathbf{a} \in RS(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n)$;
- 3 If \mathbf{a} is a total degree and $\mathbf{a} \leq \mathbf{q}$, then $\mathbf{a} \in CRS(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n)$.



Quasi-Minimal Degree

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Theorem

For any structures $\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n$ there exists an enumeration degree \mathbf{q} such that:





- 1 $\mathbf{q} \notin CRS(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n)$;
- 2 If \mathbf{a} is a total degree and $\mathbf{a} \geq \mathbf{q}$, then $\mathbf{a} \in RS(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n)$;
- 3 If \mathbf{a} is a total degree and $\mathbf{a} \leq \mathbf{q}$, then $\mathbf{a} \in CRS(\mathfrak{A}, \mathfrak{A}_1 \dots \mathfrak{A}_n)$.



Relative degree spectra

- The **Minimal pair theorem**.
- The **Quasi-minimal degree**.
- Questions:
 - Find other specific properties of Relative spectra of structures?
 - For any structures $\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n$, does there exist a structure \mathfrak{B} such that $DS(\mathfrak{B}) = RS(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n)$?



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