On the complexity of the Sperner Lemma

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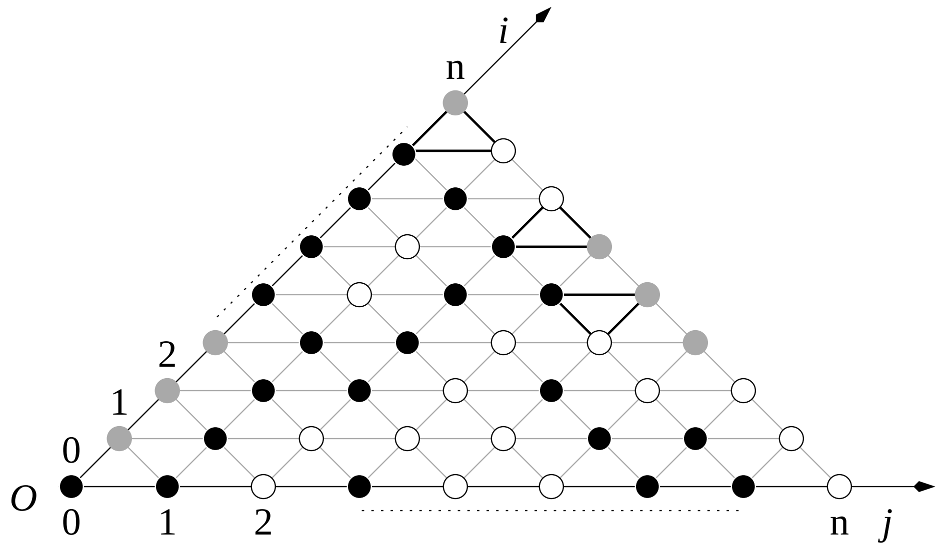
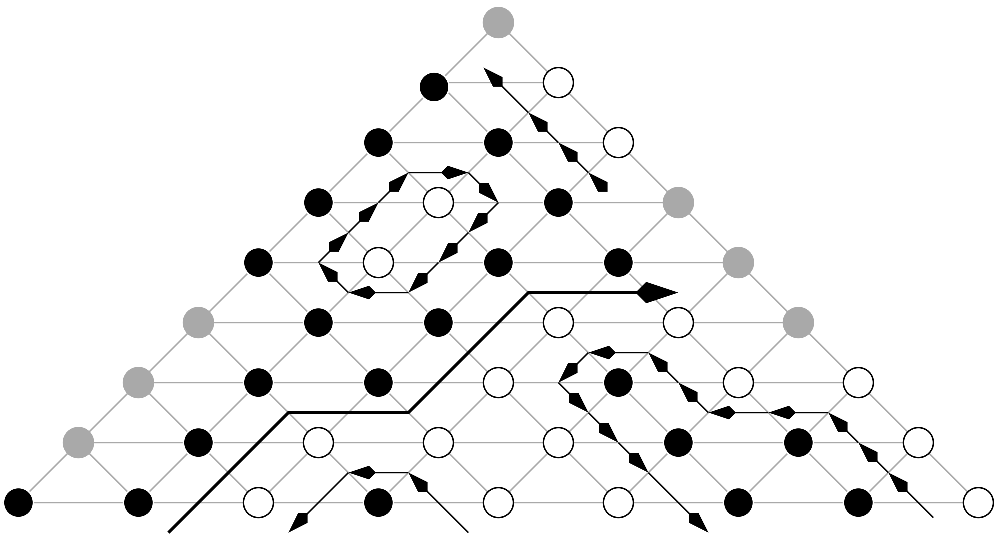
Sperner Lemma: reminder

- Combinatorial statement which can be used to prove the Fixed-Point Theorem(s)
- Proven by E. Sperner in 1928
- Very nice combinatorial proof - path following parity argument



Sperner Lemma: proof



Computational Complexity point of view: questions

1. Proof Complexity: consider a propositional encoding $Sperner_n$, what is the size of the shortest proof in a given propositional proof system (say bounded-depth Frege)?

2. Black-box (Query) Complexity: how many queries are needed in order to find a trichromatic triangle in $Sperner_n$ (known lower bounds: $\Omega(n)$ deterministic, $\Omega(\sqrt[4]{n})$ randomised, and $\Omega(\sqrt[8]{n})$ quantum)?

Propositional Encoding $Sperner_n$

$$
\begin{array}{ll}
s^1_{i\,j} \vee s^2_{i\,j} \vee s^3_{i\,j} & 0 \le i, j < n,\ i+j \le n \\
\neg s^p_{i\,j} \vee \neg s^q_{i\,j} & 0 \le i, j < n,\ i+j \le n,\ 1 \le p < q \le 3 \\
\neg s^3_{0\,j} & 0 \le j \le n \\
\neg s^2_{i\,0} & 0 \le i \le n \\
\neg s^1_{i\,n-i} & 0 \le i \le n \\
\neg s^p_{i\,j} \vee \neg s^q_{i\,j+1} \vee \neg s^r_{i+1\,j} & 0 \le i, j < n,\ 0 \le i+j < n,\ \{p,q,r\} = \{1,2,3\} \\
\neg s^p_{i\,j} \vee \neg s^q_{i\,j-1} \vee \neg s^r_{i-1\,j} & 0 < i, j \le n,\ 0 < i+j \le n,\ \{p,q,r\} = \{1,2,3\}
\end{array}
$$

Bijective Pigeon-Hole Principle PHP_n^{n+1}

$$\bigvee_{j=1}^{n} p_{i\,j} \qquad 1 \le i \le n+1$$

$$\neg p_{i\,j} \vee \neg p_{i\,k} \qquad 1 \le i \le n+1,\ 1 \le j < k \le n$$

$$\bigvee_{i=1}^{n+1} p_{i\,j} \qquad 1 \le j \le n$$

$$\neg p_{i\,j} \vee \neg p_{k\,j} \qquad 1 \le j \le n,\ 1 \le i < k \le n+1$$

$p_{i\,j}$ stands for “pigeon i goes to hole j”.

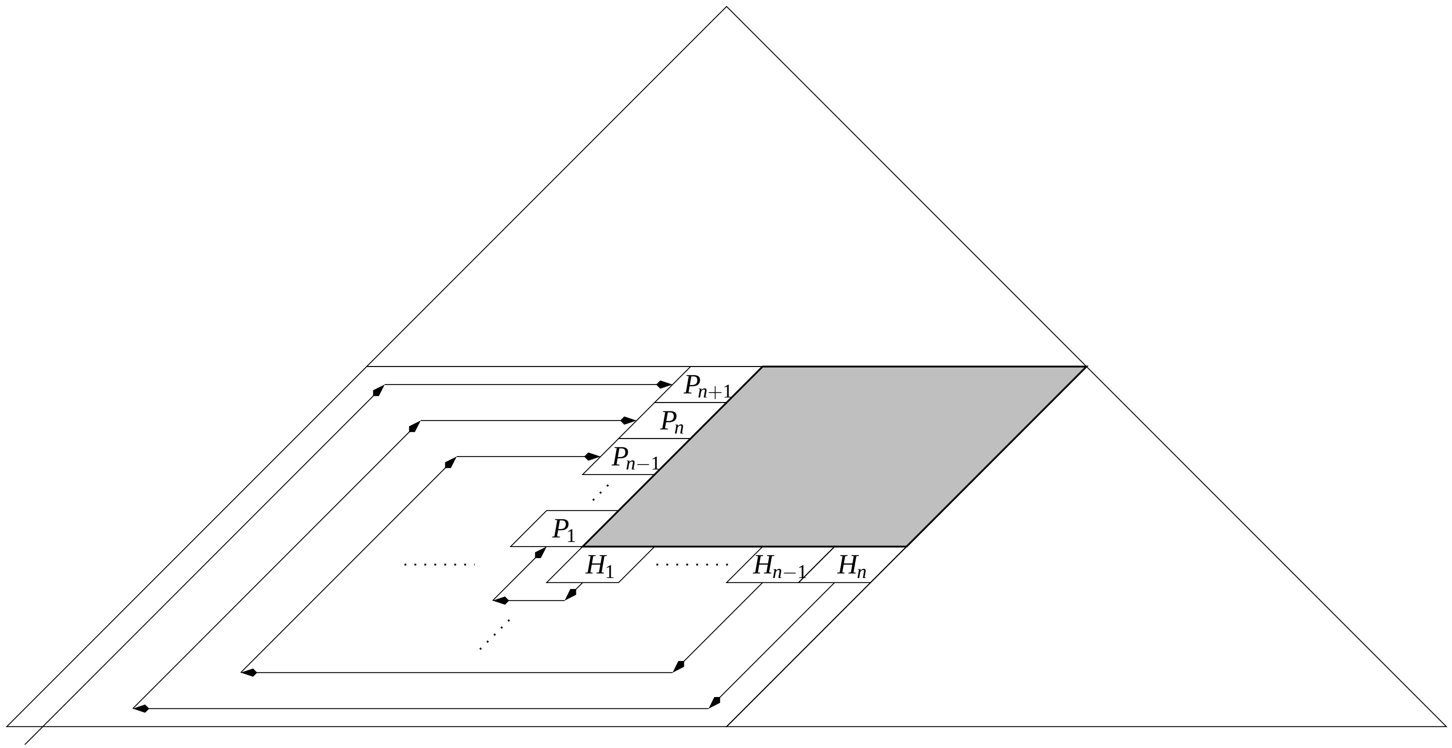
Computational Complexity point of view: our result

Proposition. *There is a reduction from* PHP_n^{n+1} *to* $Sperner_{12n+6}$ *such that every propositional variable* $s_{i\,j}^r$ *is substituted by a size* $O(n)$ *depth* 2 *formula over the variables* p *.*

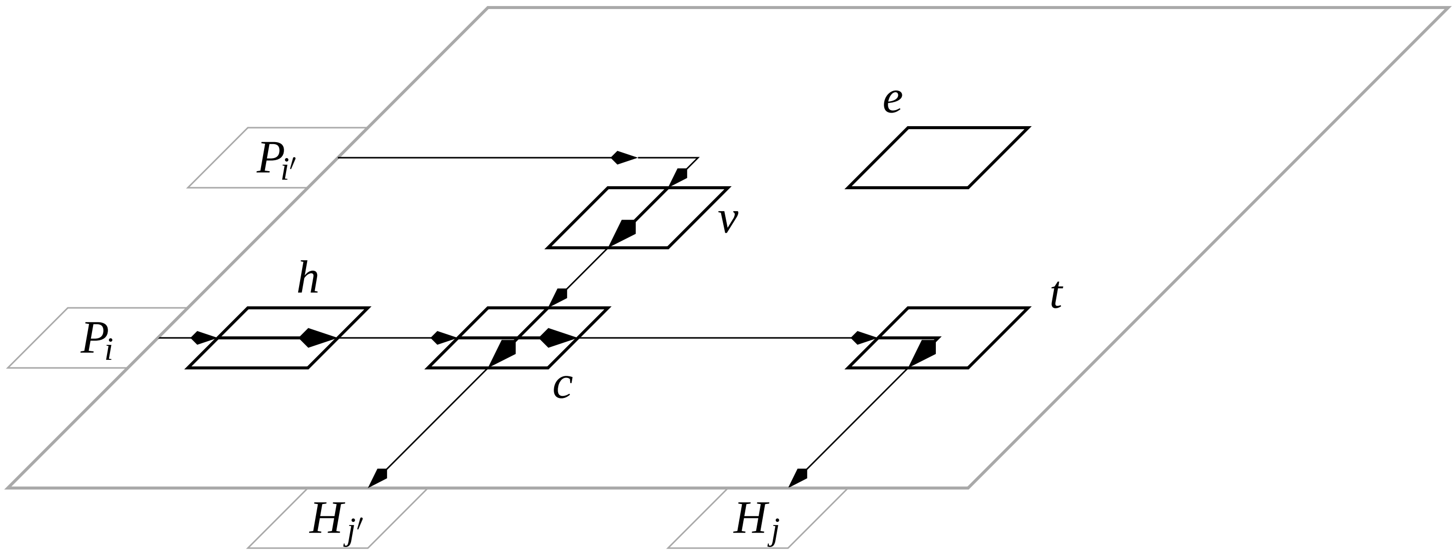
Corollary 1. *Every constant-depth Frege refutation of* $Sperner_n$ *is of size exponential in* n*.*

Corollary 2. *Every deterministic (probabilistic?) algorithm that solves the search problem for* $Sperner_n$ *has to make* $\Omega(n)$ *queries.*

Reducing PHP_n^{n+1} to $Sperner_{12n+6}$



Reducing PHP_n^{n+1} to $Sperner_{12n+6}$



Reducing PHP_n^{n+1} to $Sperner_{12n+6}$