Degrees of Weakly Computable Reals

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Left-computable reals

- We will consider only reals in the unit interval.

- A real α is *computable* (or *recursive*) if there is an algorithm deciding each digit in its binary expansion.

- Recall that a set C is computable if there is an algorithm deciding the membership of C.

- Equivalently, α is computable if there is a computable increasing sequence of rationals $\{r_n : n \in \omega\}$ and a computable function (modulus function) $g : \omega \to \omega$ such that for any m, n with $n > g(m)$, $\alpha - r_n \leq 2^{-m}$.

  We say that the sequence $\{r_n\}_{n \in \omega}$ converges to α effectively.

- In this characterization, we know how close one is to the final value. If we don't have the modulus function g, that is, the rate of convergence of $\{r_n\}_{n \in \omega}$ is controlled by g, then α can be noncomputable.

- A real α is *left-computable* if it is the limit of a computable, increasing, converging sequence of rationals $\{r_n : n \in \omega\}$.

- Recall that a set C is recursively enumerable if C is empty or C can be enumerated effectively.

  Let C be any noncomputable r.e. set. Then $0.\chi_C$ is left-computable but not computable.

- (Soare) A real α is called *strongly recursively enumerable* (s.r.e. for short) if there exists a recursively enumerable set $A \subset \omega$ such that $\alpha = \sum_{n \in A} 2^{-n}$.

- Soare proved that there are left-computable reals that are not strongly r.e..

Weakly computable reals

- The collection of left-computable reals do not behave well algebraically since, for instance, $1 - \Omega$ is not left-computable, where Ω is the halting probability of a universal prefix-free Turing machine.

- A real α is *weakly computable* if there are left-computable reals β and γ such that α equals to $\beta - \gamma$. Weakly computable reals are also called d.c.e. reals.

- (Ambos-Spies, Weihrauch and Zheng)
  - The set of weakly computable reals is closed under the arithmetic operations, and hence forms a field.
  - A real number x is weakly computable iff there is a computable sequence $\{x_s\}_{s \in \omega}$ of rational numbers which converges to x such that

    $$\sum_{s \in \omega} |x_s - x_{s+1}| \leq c$$

    for a constant c.

- This gives an analytical characterization of weakly computable reals.

Degrees of weakly computable reals

- (Zheng) There is a weakly computable real not having ω-r.e. degree.

- (Downey, Wu and Zheng)
  - Every ω-r.e. degree contains a weakly computable real.
  - There is a degree below $\mathbf{0}'$ containing no weakly computable reals.

- (Downey) There is a nonzero r.e. degree such that every degree below it contains a weakly computable real.
  - Proof using the existence of strongly contiguous (r.e.) degrees.

Nonbounding degrees

- (Ng, Stephan and Wu) There is a degree below $\mathbf{0}'$ such that every nonzero degree below it contains no weakly computable reals.

  We call such degrees *nonbounding* degrees.

- Requirements:
  - $\mathcal{R}_e$: Φ_e^A total $\Rightarrow$ Φ_e^A is computable or there is an ε such that $|\Phi_e^A - (\alpha - \beta)| > \varepsilon$.
  - $\mathcal{P}_e$: A incomputable.

- We use a variant of the standard e-splittings to satisfy the $\mathcal{R}$-requirements and use $\mathbf{0}'$ as our oracle.

- Such nonbounding degrees can be 1-generic. However, not every 1-generic degree can be nonbounding.

Yates degrees

- (Yates, 1967) There are degrees $\mathbf{d}$ below $\mathbf{0}'$ such that the r.e. degrees comparable with $\mathbf{d}$ are exactly $\mathbf{0}$ and $\mathbf{0}'$.

- Yates' degree $\mathbf{d}$ can be 1-generic (low), and can also be minimal (low_2).

  The proofs use the existence of noncuppable degrees and the fact that every nonzero r.e. degree has a 1-generic complement (Slaman and Steel) and a minimal complement (Seetapun and Slaman).

- (Wu, 2006) Yates' degrees appear in every jump class.

  Again, the proof uses the existence of noncuppable degrees.

f-limit-genericity

- A set A is called f-limit-generic iff for each $e \in \mathbb{N}$, if there are infinitely many m such that $W^K_{e,f(m)}$ contains an extension of $A(0)A(1)\cdots A(m)$, then there is an n such that W^K_e contains $A(0)A(1)\cdots A(n)$.

- A f-limit-generic set A forces membership in W^K_e only if for infinitely many prefixes

$$A(0)A(1)\cdots A(m)$$

  of A an extension in W^K_e can be found within time $f(m)$.

- f-limit-genericity differs from the 1-genericity by having an oracle and the search bound.

- If f is growing fast enough, then f-limit-genericity implies 1-genericity.

- (Ng, Stephan and Wu) Assume that $\{\alpha_i\}_{i\in\mathbb{N}}$ is a list of weakly computable reals such that there is a K-recursive function $g : \mathbb{N}\times\mathbb{N}\to\mathbb{Q}$ with $\forall i,j\,(|\alpha_i - g(i,j)| < 2^{-j})$. Then there is a function $f \leq_T K$ such that:

  (1) every f-limit-generic set A is 1-generic,

  (2) for all i, if $\alpha_i \leq_T A$, then α_i is recursive,

  (3) for all i, if $\alpha_i \geq_T A$, then α_i is complete.

  (4) Furthermore, one can choose A such that $A \leq_T K$ and hence A can be low.

- (Ng, Stephan and Wu) We know that there is an enumeration $\{\alpha_i\}_{i\in\mathbb{N}}$ of *all weakly computable reals* and a function g as above,

  so immediately, we have:

  – There is a degree $\mathbf{a}$ below $\mathbf{0}'$ such that the degrees containing weakly computable reals which are comparable with $\mathbf{a}$ are exactly $\mathbf{0}$ and $\mathbf{0}'$.

  – This generalizes Yates' result mentioned before.

- (Ng, Stephan and Wu, in progress) Such degrees also appear in every jump class.

Ω numbers and related

- Chaitin introduced Ω numbers as the halting probability of universal prefix-free machines.

- Kučera and Slaman proved that the Ω-numbers cover indeed all the left-computable Martin-Löf random sets.

- Properties of Ω:
  - Ω has a recursive approximation $\Omega_0, \Omega_1, \cdots$ from the left as it is left-computable.
  - The convergence module c_Ω defined as

    $$c_\Omega(n) = \text{min}\{s : \forall m \leq n\,(\Omega_s(m) = \Omega(m))\}$$

    dominates all total-recursive functions.
  - There are nonrecursive sets A such that Ω is random relative to A. These sets are called low for Ω.

Completely weakly computable

- A Turing degree is called *completely weakly computable* if every set in this degree is weakly computable.

- It is a topic on the other extreme.

- (Downey, Hirschfeldt, Miller and Nies) Every Δ_2^0 degree low for Ω is completely weakly computable, and such degrees can be nonrecursive.

- (Ng, Stephan and Wu) For any r.e. set A, the following are equivalent:

  1. The Turing degree of A is array recursive;
  2. Every $B \leq_T A$ is weakly computable;
  3. The Turing degree of A is completely weakly computable.

- This gives another characterization of array recursiveness, a notion first proposed by Downey, Jockusch and Stob.

Summary

In this paper, we prove the following:

1. There is a 1-generic degree below $\mathbf{0}'$ such that every nonzero degree below it contains no weakly computable reals.

2. There is a degree $\mathbf{a}$ below $\mathbf{0}'$ such that the degrees containing weakly computable reals which are comparable with $\mathbf{a}$ are exactly $\mathbf{0}$ and $\mathbf{0}'$.

3. An r.e. degree is array recursive iff it is completely weakly computable iff every set in this degree is weakly computable.