Hypercomputing the Mandelbrot Set?

at CiE 2006 in Swansea/Abertawe

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Health warning

- Do not use talk without the question mark in the title.
- There will be no stunning new results in this presentation.

The initial motivation of this paper was simply to explore the use of simple *natural* (i.e. without thinly disguised Halting Problem oracle) geometric objects as sources of 'hypercomputation'.

Instead, the paper has turned mainly into a survey of results by others about computability notions for sets in $\mathbb{R}^n$, an issue reinvigorated by the attention of Roger Penrose.

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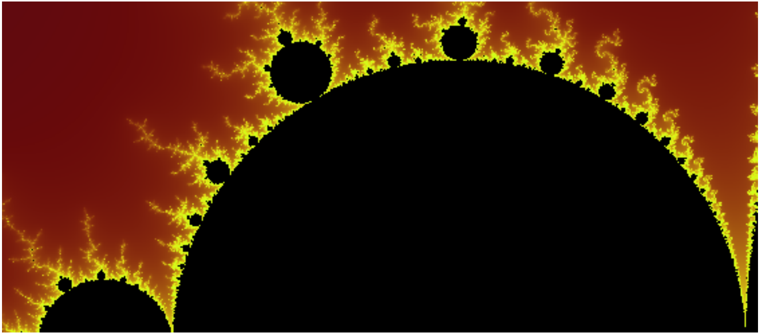
Motivation

The classical theory of computability (Church, Turing etc.) is based on the natural numbers.

- A function $f : \mathbb{N}_0 \to \mathbb{N}_0$ is *computable* if a program exists for computing all its values in finite time, but without restriction on the physical resources that may be used.
- A subset of $\mathbb{N}_0$ is called computable when its characteristic function is.
- The notion of *computable* real function, as used in computable analysis, is also quite well understood and accepted.

What is the appropriate corresponding notion for a Euclidean set?

What do we see here?



Plot done using Yannick Gingras' Fract (http://fract.ygingras.net/).

Definition of the Mandelbrot set

Answer to the Z$1m question: nothing really.

Consider $f_c(x) = x^2 + c$. In 1979 Benoît Mandelbrot used a computer to generate a plot [6] resembling a set first studied by Fatou,

$$M = \{c \in \mathbb{C} \mid \text{for all } n \geq 1,\ |f_c^n(0)| \leq 2\} \tag{1}$$

and now called the Mandelbrot set.

The usual graphic representation of M takes a fixed n and just plots

$$M_n = \{c \in \mathbb{C} \mid |f_c^n(0)| \leq 2\}\,. \tag{2}$$

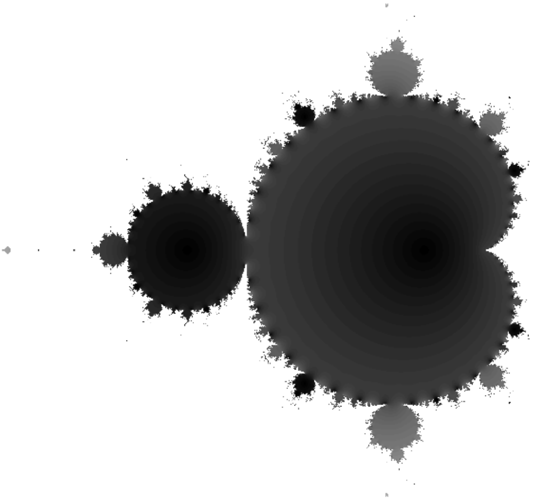
Using Octave

```
# Plots a monochrome Mandelbrot set, kind of, in Octave or
# (probably)  also in MATLAB.

n=1000;                                # For an nxn grid
m=50;                                  # Number of iterations
c=meshgrid(linspace(-2,2,n))\          # Set up grid
  +i*meshgrid(linspace(2,-2,n))';
x=zeros(n,n);                          # Initial value on grid
for i=1:m
   x=x.^2+c;                           # Iterate the mapping
endfor
imagesc(min(abs(x),2.1))               # Plot monochrome, absolute
                                       # value of 2.1 is 'escape'
```

Points coloured white have already escaped after 50 iterations.

Gray and black points are *candidates* for membership of M.



How big is M?

The A area of M is not known, in spite of pixel-count estimates,

- Tord Malmgren on a Commodore-64 in 1988: $A \approx 1.57$;
- Robert Munafo [7]: $A \approx 1.50659177$

an analytic expression that converges extremely slowly,

- John Ewing and Glenn Schober [4], 10^{1181} terms needed (see `http://mathworld.wolfram.com/MandelbrotSet.html`)to get the first three digits of $A \leq 1.7274$

and one *conjecture* (Cyril Soler, 2000, reported by Munafo) of a simple formula

$$\sqrt{6\pi - 1} - e \approx 1{,}50659165148550.$$

Everyone believes that they have seen M but apparently nobody is quite sure how big it is.

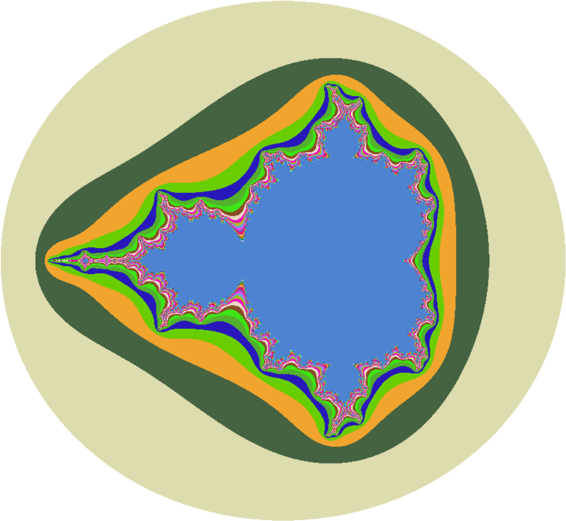
The pretty picture in Octave

```
# Plots a polychrome Mandelbrot set, kind of, in Octave or
# (probably)  also in MATLAB.

n=1000;                                # For an nxn grid
m=40;                                  # Number of iterations
c=meshgrid(linspace(-2.5,1.75,n))\     # Now set up grid
  +i*meshgrid(linspace(2.125,-2.125,n))';
x=zeros(n,n);y=ones(n,n);         # Initial values on grid
                                  # y counts number of iterations
                                  # without escape from the disk
for i=1:m
   x=x.^2+c;                      # Usual iteration
   y=y.+(.5+sign(3-abs(x))./2);# Add one if still in radius 3
   x=x.*(min(abs(x),3)\           # Scale back points far away to
     ./(abs(x)+!abs(x)));         # speed up and avoid overflow
endfor
imshow(imagesc(y),rand(m+1,3)) # Plot y with random colours
```

M in blue

Membership candidates are in the blue area in the centre. Outer coloured bands have already escaped.



Penrose's question

More than 15 years ago already, Penrose asked whether M is computable/decidable in any sense. Since there is no canonical concept of computability of sets in the plane (unlike for subsets of the natural numbers), an answer to the question would first require a suitable formulation of the notion.

This question can basically be tackled in two ways.

1. Investigate the existence of an algorithmic procedure that can determine, on input x, whether $x \in M$ or not. The notion *algorithmic* and the range of the possible inputs x must be clear delimited (Blum-Shub-Smale, rational points in this talk).
2. Redefine the notion otherwise (Weihrauch, Brattka, Hertling and others).

Four Approaches & Penrose's Criterion

Four computability concepts for sets in the plane will be briefly considered:

- restriction to rational points;
- to recursive points;
- B-S-S computation;
- and the approach of computable analysis.

Look at the Mandelbrot set M as well as the unit disk

$$D = \{z \in \mathbb{C} \mid |z| \leq 1\}$$

and the epigraph of the exponential function

$$E = \{x + iy \in \mathbb{C} \mid y \geq e^x\}$$

in each of the four approaches.

Penrose's Criterion: any reasonable notion of *computability* would call the sets D and E computable.

Computability w.r.t. the rational points

A naïve approach is for $X \subseteq \mathbb{C}$ to be called $\mathbb{Q}$-*computable* if its restriction to points with rational real and imaginary parts is computable in the Church-Turing sense via a suitable $f : \mathbb{N}_0^6 \longrightarrow \{0;1\}$ such that

$$f(k;m;n;\ell;p;q) = 1 \quad \text{iff } pn \neq 0 \text{ and } \quad (-1)^k \frac{m}{n} + i(-1)^\ell \frac{p}{q} \in X.$$

D and E are of course $\mathbb{Q}$-computable (and we need the fact that e^x is irrational for every non-zero rational x) but it is not known (to the speaker) whether M is.

No disk around 0 with non-computable radius is $\mathbb{Q}$-computable, of course!

The Markov approach

A real number x is computable if a classical computable function exists that produces rational approximations to x.

Identify x with such a programme i_x. A function $f : \mathbb{R}_c \rightarrow \mathbb{R}_c$ on the computable reals is called Markov-computable if there exists a programme transforming each input i_x into output $i_{f(x)}$ in finite time.

A subset of $\mathbb{C}$ can be called Markov-computable if its characteristic function is Markov-computable.

However, all Markov-computable functions are continuous, so the only Markov-computable sets are the empty set and the entire plane.

Blum-Shub-Smale real machines

The BSS computational model [2, 1] is defined over any commutative ring (including $\mathbb{R}$) for which the classical computability is a special case where the ring is $\mathbb{Z}_2$.

The basic idea is that a B-S-S machine can, at least, perform the ring operations in a single step which—apparently—corresponds to the thinking of some numerical analysts.

The Mandebrot set M was shown by Blum and Smale to be non-computable using their real machines. However, Vasco Brattka has shown [3] that neither is E, the epigraph of the exponential function.

B-S-S computability does not, therefore, satisfy the Penrose Criterion.

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Computable analysis and the distance function

A function is computable if it maps the set of computable sequences of computable reals to itself and is computably continuous (in computable analysis, from the Polish school).

Weihrauch, Hertling, Brattka e.a. call those sets of the plane *computable* which have computable metric distance functions.

Their approach preserves the classical notion of a recursive subset of the natural numbers, when the natural numbers are embedded in the plane in the natural way.

It is not yet known whether M has a computable distance function but Peter Hertling [5] has shown that if certain topological conditions are satisfied (and they would be implied by locally connectedness of M) then M would be computable in this approach.

Zeno-type machines

Using a hypercomputational device which can complete countable many steps in finite time, all three sets are decidable w.r.t. the rational points, of course.

```
n=1000;                          # For an nxn grid
c=meshgrid(linspace(-2,2,n))\   # Set up grid
+i*meshgrid(linspace(2,-2,n))';
x=zeros(n,n);                    # Initial value on grid
do
   x=x.^2+c;                     # Usual iteration
   x=x.*(min(abs(x),3)\          # Scale back points far away to
    ./(abs(x)+!abs(x)));         # speed up and avoid overflow
                                 # and infinite values
until (1==0)                     # Repeat a lot
imagesc(min(abs(x),2.1))         # Plot x, 2.1 counts as escape
```

This approach really looks like overkill, of course, and is well-illuminated by flickering lamps.

Computable?

The results discussed so far, in summary:

	D	E	M
Markov approach	$\times$	$\times$	$\times$
Blum-Shub-Smale machines	$\checkmark$	$\times$	$\times$
$\mathbb{Q}$*-computability*	$\checkmark$	$\checkmark$	**?**
Computable analysis (Brattka e.a.)	$\checkmark$	$\checkmark$	**?**
Zeno-type machines	$\checkmark$	$\checkmark$	$\checkmark$

We close with examples that easily show that $\mathbb{Q}$-computability and the computable analysis approaches are distinct, as one would expect.

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We close with examples that easily show that $\mathbb{Q}$-computability and the computable analysis approaches are distinct, as one would expect.

Are $\mathbb{Q}$-computability and CA-computability really different?

Let (x_{nm}) be a lower-triangular binary matrix, increasing in each column. That is, for all n, m

- $x_{nm} \in \{0,1\}$;
- $x_{nm} = 0$ whenever $m > n$; and
- $x_{nm} \leq x_{nm+1}$.

Set

$$a_n = \sum_{m=1}^{\infty} \frac{x_{nm}}{2^m}.$$

Now, $a_n \to a \leq 1$ of course, since the sequence (a_n) is increasing and bounded.

We will use the following fact: there exists a *computable* assignment $(n,m) \mapsto x_{nm}$ such that the a obtained as above is a non-computable real number. Fix one such assignment.

Example: The Funnel

Using the sequence $a_n \to a \leq 1$ of the previous slide, set

$$F = \bigcup_{n=1}^{\infty} \left\{ (x,y) \,\middle|\, \frac{1}{n+1} \leq x \leq \frac{1}{n}; a_n \leq y \leq a_{n+2} \right\} \bigcup \{(0,a)\}.$$

For any $(q,r) \in \mathbb{Q}^2$ it is easy to determine whether $(q,r) \in F$ using the algorithm for computing x_{mn}, of course.

The distance to F is, however, not computable since the distance from $(0,1)$ to F would be (i) computable and (ii) equal to $1 - a$, which is a contradiction.

This shows that $\mathbb{Q}$-computability does not necessarily imply CA-computability.

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Example: The Step Set

Using the same matrix (x_{nm}) as before, set

$$b_m = \sum_{n=m}^{\infty} \frac{1 - x_{nm}}{2^{n-m+1}}$$

and define the Step Set,

$$S = \cup_{m=1}^{\infty} [m, m+1] \times [0, b_m].$$

Now, the distance function to S is computable, but there exists no algorithmic procedure for determining membership of S for even the points $\left(m + \frac{1}{2}, 1\right)$—some of which belong the set—since such a procedure would allow is to determine a to an arbitrary degree of accuracy—contradicting the non-computability of a.

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Directions for future research

- Try to show that the Mandelbrot set is CA-computable or $\mathbb{Q}$-computable.
- If M is not $\mathbb{Q}$-computable, can the Halting Problem be reduced to determining membership of $M \cap \mathbb{Q}^2$, i.e. how powerful a 'hypercomputer' is the Mandelbrot set?
- Find a reasonable class of sets for which the notions of CA- and $\mathbb{Q}$-computable coincide, if possible.
- Propose another reasonable computability notion for Euclidean sets.
- Establish a list of plausible criteria which any such notion should satisfy.

A word of caution: computability as a very physical or geometric notion is inherently problematic for well-known reasons such as the lack of scale invariance.

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