A Measure of Space for Computing over the Reals

Paulin Jacobé de Naurois

LIPN - Université Paris XIII

CIE 2006 - Swansea - Wales

BSS model over the Reals

- Arithmetic and comparison gates with unbounded precision and unit cost over real numbers
- Sequential and parallel complexity classes: $P_{\mathbb{R}}, NP_{\mathbb{R}}, NC_{\mathbb{R}}$, ...
- Michaux's Result - Computing in Constant Unit Space -> No space complexity classes
- Koiran's Weak Model -> weak measure of time

Questions of Space

Unit measure of space: # of tape cells used.

Michaux's Result
Let $L \subseteq \mathbb{R}^*$ be a language decided in bounded time by a machine M. There exists $k \in \mathbb{N}$ and a machine M' deciding L in bounded time and working space less than k.

Motivation of this Work

Let M be a boolean algorithm in LOGSPACE

Motivation of this Work

Let M be a boolean algorithm in LOGSPACE
Let M' be a real algorithm, such that:

- M' reads $(x_1, \ldots, x_n) \in \mathbb{R}^n$.
- M' computes $(\sigma_1, \ldots, \sigma_n) \in \{0,1\}^n$,

$$\sigma_i = \begin{cases} 1 & \text{if } x_i \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

- M' applies M on input $(\sigma_1, \ldots, \sigma_n)$

Motivation of this Work

Let M be a boolean algorithm in LOGSPACE
Let M' be a real algorithm, such that:

- M' reads $(x_1, \ldots, x_n) \in \mathbb{R}^n$.
- M' computes $(\sigma_1, \ldots, \sigma_n) \in \{0,1\}^n$,

$$\sigma_i = \begin{cases} 1 & \text{if } x_i \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

- M' applies M on input $(\sigma_1, \ldots, \sigma_n)$

$M' \in \mathsf{NC}^2_{\mathbb{R}}$.

Motivation of this Work

Let M be a boolean algorithm in LOGSPACE
Let M' be a real algorithm, such that:

- M' reads $(x_1, \dots, x_n) \in \mathbb{R}^n$.
- M' computes $(\sigma_1, \dots, \sigma_n) \in \{0,1\}^n$,

$$\sigma_i = \begin{cases} 1 & \text{if } x_i \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

- M' applies M on input $(\sigma_1, \dots, \sigma_n)$

$M' \in \mathsf{NC}^2_{\mathbb{R}}$. Existence of a more natural class?

Michaux's Result: Computed Values

Let M be a machine, with

- m constant nodes, $A_1, \ldots, A_m \in \mathbb{R}^m$.
- a bound $t(n)$ on the computation time, for inputs of size n.

Michaux's Result: Computed Values

Let M be a machine, with

- m constant nodes, $A_1, \ldots, A_m \in \mathbb{R}^m$.
- a bound $t(n)$ on the computation time, for inputs of size n.

Lemma:
On any input $x_1, \ldots, x_n \in \mathbb{R}^n$, at any computation step k, any non-empty cell e_l on the work tape holds the evaluation of a rational fraction $f_{l,k} \in \mathbb{Z}(X_1, \ldots, X_{n+m})$ on $(x_1, \ldots, x_n, A_1, \ldots, A_m)$.

Michaux's Result: Simulation

Step k: $y_{l,k} = f_{l,k}(x_1, \ldots, x_n, A_1, \ldots, A_m)$.

Michaux's Result: Simulation

Step k: $y_{l,k} = f_{l,k}(x_1, \ldots, x_n, A_1, \ldots, A_m)$.

$y_{l,k}$ can be represented by $\overline{y_{l,k}} \in \{0,1\}^*$: binary representation of $f_{l,k}$.

Michaux's Result: Simulation

Step k: $y_{l,k} = f_{l,k}(x_1, \ldots, x_n, A_1, \ldots, A_m)$.

$y_{l,k}$ can be represented by $\overline{y_{l,k}} \in \{0,1\}^*$: binary representation of $f_{l,k}$.

The work tape can be represented by $(\overline{w_l}, \overline{w_r}) \in (\{0,1\}^*)^2$: binary representation of the left and right parts.

Michaux's Result: Simulation

Step k: $y_{l,k} = f_{l,k}(x_1, \ldots, x_n, A_1, \ldots, A_m)$.

$y_{l,k}$ can be represented by $\overline{y_{l,k}} \in \{0,1\}^*$: binary representation of $f_{l,k}$.

The work tape can be represented by $(\overline{w_l}, \overline{w_r}) \in (\{0,1\}^*)^2$: binary representation of the left and right parts.

The work tape can be represented by $(c_l, c_r) \in \mathbb{R}^2$: numerical values for $(\overline{w_l}, \overline{w_r})$.

Michaux's Result: Simulation

Step k: $y_{l,k} = f_{l,k}(x_1, \ldots, x_n, A_1, \ldots, A_m)$.

$y_{l,k}$ can be represented by $\overline{y_{l,k}} \in \{0,1\}^*$: binary representation of $f_{l,k}$.

The work tape can be represented by $(\overline{w_l}, \overline{w_r}) \in (\{0,1\}^*)^2$: binary representation of the left and right parts.

The work tape can be represented by $(c_l, c_r) \in \mathbb{R}^2$: numerical values for $(\overline{w_l}, \overline{w_r})$.

Simulation of one arithmetical computation step: symbolic binary computation.

Michaux's Result: Simulation

Step k: $y_{l,k} = f_{l,k}(x_1, \ldots, x_n, A_1, \ldots, A_m)$.

$y_{l,k}$ can be represented by $\overline{y_{l,k}} \in \{0,1\}^*$: binary representation of $f_{l,k}$.

The work tape can be represented by $(\overline{w_l}, \overline{w_r}) \in (\{0,1\}^*)^2$: binary representation of the left and right parts.

The work tape can be represented by $(c_l, c_r) \in \mathbb{R}^2$: numerical values for $(\overline{w_l}, \overline{w_r})$.

Simulation of one arithmetical computation step: symbolic binary computation.

Simulation of one branch step: numerical evaluation of the arguments, and comparison.

Koiran's Weak Model

- Unit measure of time $\rightarrow$ not realistic enough?
- Weak measure of time: a repeated sequence of additions or multiplications has an increasing cost

Koiran's Weak Model

- Unit measure of time $\rightarrow$ not realistic enough?
- Weak measure of time: a repeated sequence of additions or multiplications has an increasing cost

Ex: $y * z$

$$
\begin{aligned}
y &\longrightarrow f_y \in \mathbb{Z}(x_1, \ldots, x_n, A_1, \ldots, A_m) \\
z &\longrightarrow f_z \in \mathbb{Z}(x_1, \ldots, x_n, A_1, \ldots, A_m)
\end{aligned}
$$

Koiran's Weak Model

- Unit measure of time $\to$ not realistic enough?
- Weak measure of time: a repeated sequence of additions or multiplications has an increasing cost

Ex: $y * z$

$$\begin{aligned} y &\longrightarrow f_y \in \mathbb{Z}(x_1, \ldots, x_n, A_1, \ldots, A_m) \\ z &\longrightarrow f_z \in \mathbb{Z}(x_1, \ldots, x_n, A_1, \ldots, A_m) \end{aligned}$$

Weak cost of $y * z$: max of the degrees and of the coefficient heights of f_y and f_z.

Complexity (4)

Lemma: $L \in \mathsf{P}_W$ if and only if:

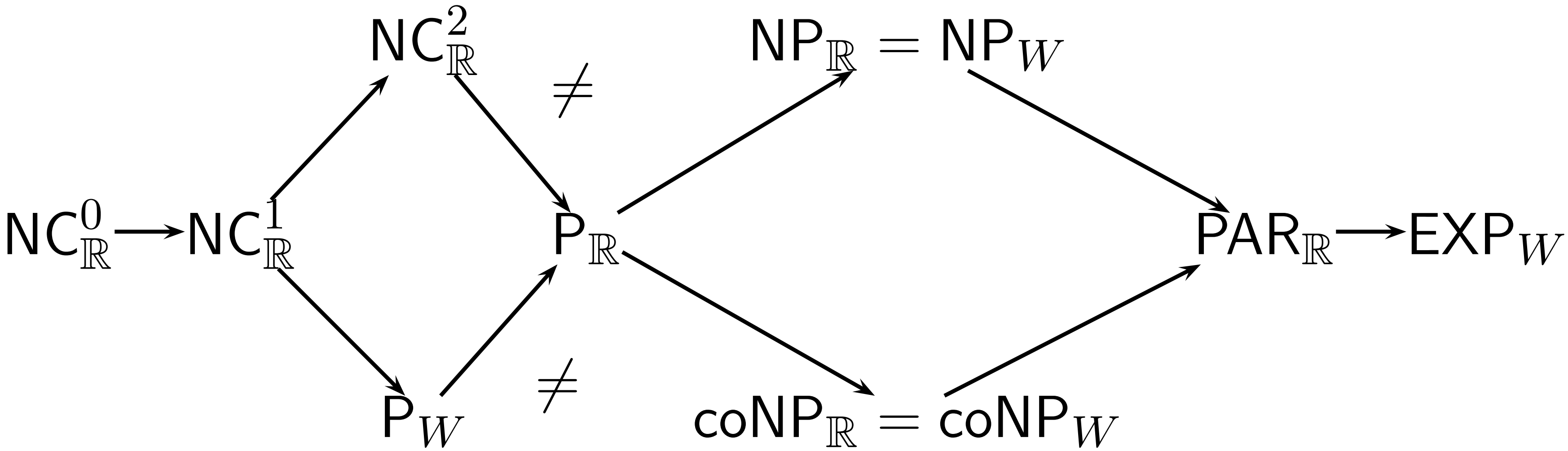
- $L \in \mathsf{P}_{\mathbb{R}}$, and
- Every computed rational fraction has polynomial degree and coefficient heights.

Complexity (4)

Lemma: $L \in \mathsf{P}_W$ if and only if:

- $L \in \mathsf{P}_{\mathbb{R}}$, and
- Every computed rational fraction has polynomial degree and coefficient heights.

Inclusions:



Motivation of this Work (2)

Let M be a boolean algorithm in LOGSPACE
Let M' be a real algorithm, such that:

- M' reads $(x_1, \ldots, x_n) \in \mathbb{R}^n$.
- M' computes $(\sigma_1, \ldots, \sigma_n) \in \{0, 1\}^n$,

$$\sigma_i = \begin{cases} 1 & \text{if } x_i \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

- M' applies M on input $(\sigma_1, \ldots, \sigma_n)$

Motivation of this Work (2)

Let M be a boolean algorithm in LOGSPACE
Let M' be a real algorithm, such that:

- M' reads $(x_1, \ldots, x_n) \in \mathbb{R}^n$.
- M' computes $(\sigma_1, \ldots, \sigma_n) \in \{0,1\}^n$,

$$\sigma_i = \begin{cases} 1 & \text{if } x_i \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

- M' applies M on input $(\sigma_1, \ldots, \sigma_n)$

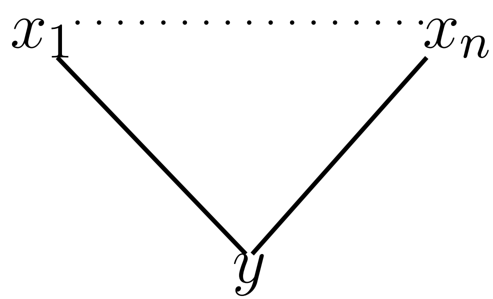
$M' \in \mathsf{NC}^2_{\mathbb{R}} \cap \mathsf{P}_W$
$M' \notin \mathsf{NC}^1_{\mathbb{R}}$?

A Weak Measure of Space

$$y \longrightarrow f_y \in \mathbb{Z}(x_1, \ldots, x_n, A_1, \ldots, A_m)$$

A Weak Measure of Space

$$y \rightarrow f_y \in \mathbb{Z}(x_1, \ldots, x_n, A_1, \ldots, A_m)$$

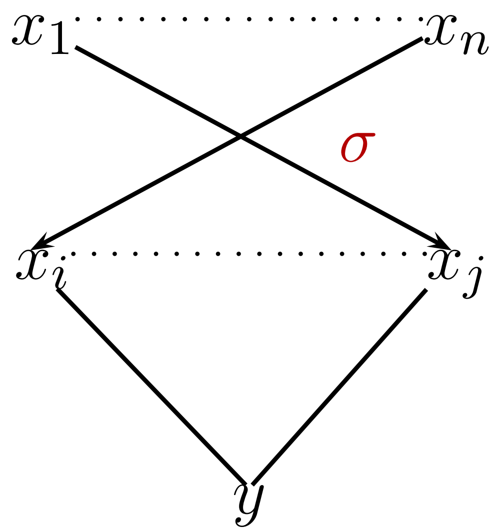


Syntactic Tree T_y of f_y

Original Idea: $|y| \sim |T_y|$.

A Weak Measure of Space

$$y \to f_y \in \mathbb{Z}(x_1, \ldots, x_n, A_1, \ldots, A_m)$$



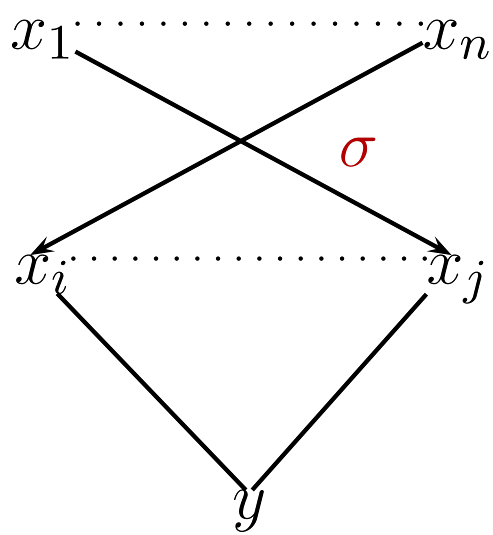
Syntactic Tree T_y of f_y

Original Idea: $|y| \sim |T_y|$.

PB 1: encoding of a permutation $\sigma : \{1, \ldots n\} \to \{1, \ldots n\}$.

A Weak Measure of Space

$$y \longrightarrow f_y \in \mathbb{Z}(x_1, \dots, x_n, A_1, \dots, A_m)$$



Syntactic Tree T_y of f_y

Original Idea: $|y| \sim |T_y|$.

PB 1: encoding of a permutation $\sigma : \{1, \dots n\} \rightarrow \{1, \dots n\}$.

PB 2: computation of a minimal tree $\rightarrow$ factorization problem.

Weak Size of a Number

$$y \to f_y \in \mathbb{Z}(x_1, \ldots, x_n, A_1, \ldots, A_m)$$

Weak Size of a Number

$y \longrightarrow f_y \in \mathbb{Z}(x_1, \ldots, x_n, A_1, \ldots, A_m)$
We restrict ourselves to circular permutations $\sigma : \{1, \ldots n\} \rightarrow \{1, \ldots n\}$. A circular permutation can be represented by an offset of size $\log(n)$.

Weak Size of a Number

$y \longrightarrow f_y \in \mathbb{Z}(x_1, \ldots, x_n, A_1, \ldots, A_m)$
We restrict ourselves to circular permutations $\sigma : \{1, \ldots n\} \to \{1, \ldots n\}$. A circular permutation can be represented by an offset of size $\log(n)$.

$|y|_W$ is the size of a explicit (sequence of monomials) boolean description of f_y, modulo the permutation.

Weak Size of a Configuration

The permutation σ is common for all numbers on the tape $\longrightarrow$ additive constant of size $\log(n)$.

Weak Size of a Configuration

The permutation σ is common for all numbers on the tape $\longrightarrow$ additive constant of size $\log(n)$.

The weak size of a configuration is the minimum for all circular permutations of the sum of the weak sizes of the numbers on the tape.

Weak Size of a Configuration

The permutation σ is common for all numbers on the tape $\rightarrow$ additive constant of size $\log(n)$.

The weak size of a configuration is the minimum for all circular permutations of the sum of the weak sizes of the numbers on the tape.

- A constant A_i has weak size 1.

Weak Size of a Configuration

The permutation σ is common for all numbers on the tape $\rightarrow$ additive constant of size $\log(n)$.

The weak size of a configuration is the minimum for all circular permutations of the sum of the weak sizes of the numbers on the tape.

- A constant A_i has weak size 1.
- An integer $k \in \mathbb{N}$ has weak size $\log(k)$.

Weak Size of a Configuration

The permutation σ is common for all numbers on the tape $\rightarrow$ additive constant of size $\log(n)$.

The weak size of a configuration is the minimum for all circular permutations of the sum of the weak sizes of the numbers on the tape.

- A constant A_i has weak size 1.
- An integer $k \in \mathbb{N}$ has weak size $\log(k)$.
- The algorithm M' is in $\mathsf{LOGSPACE}_W$.

Weak Size of a Configuration

The permutation σ is common for all numbers on the tape $\rightarrow$ additive constant of size $\log(n)$.

The weak size of a configuration is the minimum for all circular permutations of the sum of the weak sizes of the numbers on the tape.

- A constant A_i has weak size 1.
- An integer $k \in \mathbb{N}$ has weak size $\log(k)$.
- The algorithm M' is in $\mathsf{LOGSPACE}_W$.
- Michaux's simulation of M' is in $\mathsf{LOGSPACE}_W$.

Weak Size of a Configuration

The permutation σ is common for all numbers on the tape $\rightarrow$ additive constant of size $\log(n)$.

The weak size of a configuration is the minimum for all circular permutations of the sum of the weak sizes of the numbers on the tape.

- A constant A_i has weak size 1.
- An integer $k \in \mathbb{N}$ has weak size $\log(k)$.
- The algorithm M' is in LOGSPACE_W.
- Michaux's simulation of M' is in LOGSPACE_W.
- There exists some problems decidable in bounded time, not decidable in constant weak space.

$P_{\mathbb{R}}$-completeness

Theorem: $RCDP_{\mathbb{R}}$ (Real Circuit Decision Procedure) is $P_{\mathbb{R}}$-complete under $LOGSPACE_W$ reductions.

$\mathsf{P}_{\mathbb{R}}$-completeness

Theorem: $\mathsf{RCDP}_{\mathbb{R}}$ (Real Circuit Decision Procedure) is $\mathsf{P}_{\mathbb{R}}$-complete under $\mathsf{LOGSPACE}_W$ reductions.

Proof: P-completeness of the Boolean Circuit Decision Problem under $\mathsf{LOGSPACE}$-reductions.

Structural Complexity

Theorem:

- $\mathsf{LOGSPACE}_W \subset \mathsf{P}_W \cap \mathsf{NC}^2_{\mathbb{R}}$.
- $\mathsf{PSPACE}_W \subset \mathsf{PAR}_{\mathbb{R}}$.

Structural Complexity

Theorem:

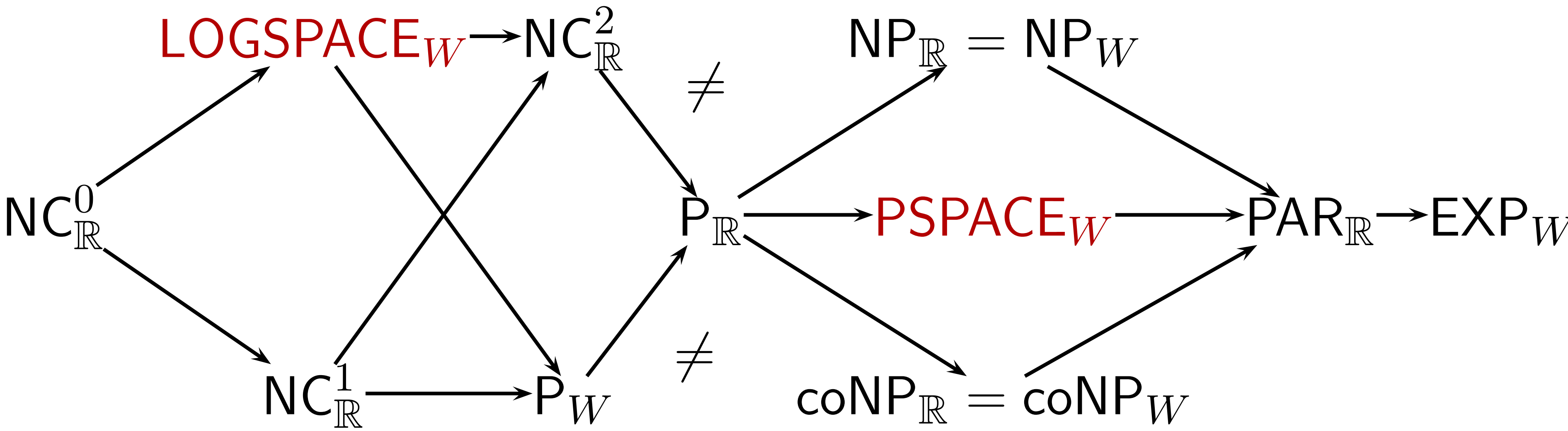
- $\mathsf{LOGSPACE}_W \subset \mathsf{P}_W \cap \mathsf{NC}^2_{\mathbb{R}}$.
- $\mathsf{PSPACE}_W \subset \mathsf{PAR}_{\mathbb{R}}$.

Proof:

- $\mathsf{LOGSPACE}_W \subset \mathsf{P}_{\mathbb{R}}$: enumeration of all configurations.
- $\mathsf{LOGSPACE}_W \subset \mathsf{P}_W$: Koiran's Lemma.
- $\mathsf{LOGSPACE}_W \subset \mathsf{NC}^2_{\mathbb{R}}$: $\mathsf{P}_{\mathbb{R}}$-uniform construction of the configuration graph of a $\mathsf{LOGSPACE}_W$ machine, and graph reachability in NC^2.
- $\mathsf{PSPACE}_W \subset \mathsf{PAR}_{\mathbb{R}}$: Corollary.

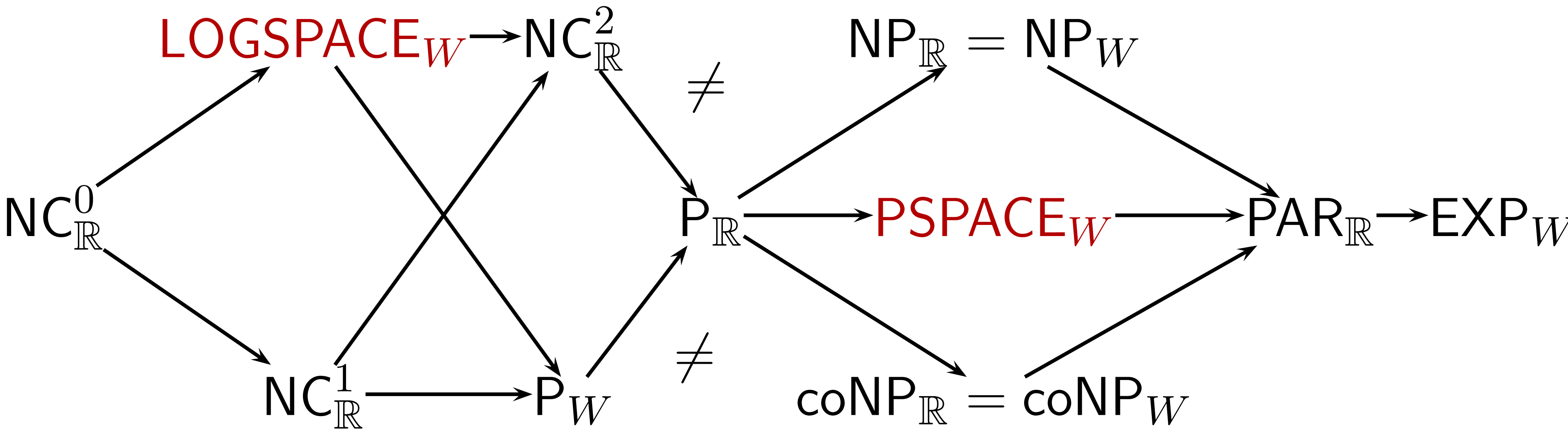
Summary and Open Questions

Inclusions:



Summary and Open Questions

Inclusions:



Conjectures:

- $\mathrm{NC}^1_{\mathbb{R}} \not\subset \mathrm{LOGSPACE}_W$, $\mathrm{LOGSPACE}_W \not\subset \mathrm{NC}^1_{\mathbb{R}}$.
- $\mathrm{PSPACE}_W = \mathrm{PAR}_{\mathbb{R}}$