A Structure with P = NP

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A Structure with P = NP

- Satisfiability of $\psi(Y_1,\ldots, Y_m)$ for relations on padded strings for
  $\Sigma_R = (\{a, b\}^*; \varepsilon; add_a, add_b, sub_a, sub_b; R, =)$
  - Satisfiability for special conjunctions $\psi(Y_1,\ldots, Y_m)$
  - Replacement of strings by small elements

- Some Satisfiability Problems for Σ

- $P_{\Sigma R} = NP_{\Sigma R}$ for $R = R_0$
  - The recursive definition of R_0
  - The reduction of $SAT_{\Sigma R_0}$

The structures and the formulae $\psi(Y_1,\ldots, Y_m)$

Satisfiability for relations on padded strings

Some Satisfiability Problems for Σ

$P_{\Sigma_R} = NP_{\Sigma_R}$ for $R = R_0$

Some simple structures:

$\Sigma_R = (\{a, b\}^*;\ \varepsilon;\ add_a, add_b, sub_a, sub_b;\ R, =)$, $\varepsilon \triangleq$ empty string

$add_a(s) = sa,\quad add_b(s) = sb\ ,\quad sub_a(sa) = s\ , \ldots$

We consider a special case as example:

$\psi = \psi(\mathbf{Y}) = \psi(Y_1,\ldots, Y_m) = \lambda_1 \wedge \cdot\cdot\cdot \wedge \lambda_k$ conjunction

- only free variables $Y_1,\ldots, Y_m$,
- $(=, \neq, R, \neg R)$-literals $\lambda_1 ,\ldots, \lambda_k$,
- $(\varepsilon, add_a , add_b)$-terms.

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Relations on padded strings

Satisfiability for relations on padded strings

Some Satisfiability Problems for Σ

$P_{\Sigma_R} = NP_{\Sigma_R}$ for $R = R_0$

$A = \{a, b\}^*$, $s, r \in A$

The padding of strings: $s_{dbl} = sa^{|s|} \in A$ (by doubling the length)

REL

$$= \{R \mid \forall s \exists r (R(s) \rightarrow s = (rb)_{dbl}) \ \& \ \forall i \forall j (i \leq j-3 \rightarrow R((a^i b^{j-i})_{dbl}))\}$$

$s = (rb)_{dbl} = d_1 \cdots d_k b \, aa \cdots aa$

$|r|+1$ characters; $(|r|+1) \times$ character a

$a \cdots ab \cdots bbb \, a \cdots aaa$

j characters; $j \times a$

Only padded strings

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Satisfiability of $\psi(Y_1,\ldots, Y_m)$ for relations on padded strings

Satisfiability for relations on padded strings

Some Satisfiability Problems for Σ

$P_{\Sigma_R} = NP_{\Sigma_R}$ for $R = R_0$

A **satisfiable** conjunction $\psi(Y_1,\ldots, Y_m)$, $|code(\psi)| = m$

Transfer for $R \in$ REL

The **new satisfiable** system where the conditions have the following form:

$$Y_j = [Y_i][d_1 \cdots d_k], \qquad Y_{i_1} \neq [Y_{i_2}][d_1 \cdots d_k],$$

$$[\neg]R(Y_i a^w), \qquad [\neg]R(Y_i d_1 \cdots d_{w_1} b a^{w_2})$$

$k, w, w_1 + w_2 + 1 < m, \quad i, i_1, i_2 \in I \subseteq \{1,\ldots, m\}, \quad j \notin I$

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Satisfiability of $\psi(Y_1,\ldots, Y_m)$ for relations on padded strings

Satisfiability for relations on padded strings

Some Satisfiability Problems for Σ

$P_{\Sigma_R} = NP_{\Sigma_R}$ for $R = R_0$

A **satisfiable** conjunction $\psi(Y_1,\ldots, Y_m)$, $|code(\psi)| = m$

Replace: $add_d(Y_i)$ by $Y_i d$, $\quad d_j, e_j, d, e \in \{a, b\}$

$[Y_i][d_1\cdots d_k]d =/\neq [Y_j][e_1\cdots e_l]d$

by $[Y_i][d_1\cdots d_k] =/\neq [Y_j][e_1\cdots e_l]$,

Y_j by $[Y_i][d_1\cdots d_k]$ if $Y_j = [Y_i][d_1\cdots d_k]$,

Remove: $Y_i = Y_i$, $\varepsilon = \varepsilon$, $Y_i \neq Y_i d_1\cdots d_k$, $\varepsilon \neq Y_i d_1\cdots d_k$, $k\geq 1$

$[Y_i][d_1\cdots d_k]\, d \neq [Y_j][e_1\cdots e_l]\, e$ if $d \neq e$,

$\neg R([Y_i][d_1\cdots d_k]b)$, $\neg R(Y_i d_1\cdots d_{w_1} b a^{w_2})$ if $w_1 \geq w_2$

Transfer for $R \in$ REL

The new satisfiable system:

$$Y_j = [Y_i][d_1\cdots d_k], \qquad Y_{i_1} \neq [Y_{i_2}][d_1\cdots d_k],$$

$$[\neg]R(Y_i a^w), \qquad [\neg]R(Y_i d_1\cdots d_{w_1} b a^{w_2}),$$

$k, w, w_1 + w_2 + 1 < m, \quad i, i_1, i_2 \in I \subseteq \{1,\ldots, m\}, \quad j \notin I$

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Satisfiability of $\psi(Y_1,\ldots, Y_m)$ for relations on padded strings

Satisfiability for relations on padded strings

Some Satisfiability Problems for Σ

$P_{\Sigma_R} = NP_{\Sigma_R}$ for $R = R_0$

A satisfiable conjunction $\psi(Y_1, \ldots, Y_m)$, $|code(\psi)| = m$

T

Replace: $add_e(Y_i)$ by $Y_i e$, $\qquad d_j, e_j, d, e \in \{a, b\}$

$[Y_i][d_1 \cdots d_k]d =/\neq [Y_j][e_1 \cdots e_l]d$

by $[Y_i][d_1 \cdots d_k] =/\neq [Y_j][e_1 \cdots e_l]$ (delete d) ,

Y_j by $[Y_i][d_1 \cdots d_k]$ if $Y_j = [Y_i][d_1 \cdots d_k]$,

Remove: $Y_i = Y_i$, $\varepsilon = \varepsilon$, $Y_i \neq Y_i d_1 \cdots d_k$, $\varepsilon \neq Y_i d_1 \cdots d_k$, $k \geq 1$

$[Y_i][d_1 \cdots d_k]d \neq [Y_j][e_1 \cdots e_l]e$ if $d \neq e$,

$\neg R([Y_i][d_1 \cdots d_k]b)$, $\neg R(Y_i d_1 \cdots d_{w_1} b a^{w_2})$ if $w_1 \geq w_2$

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Satisfiability of $\psi(Y_1,\ldots, Y_m)$ for relations on padded strings

Satisfiability for relations on padded strings

Some Satisfiability Problems for Σ

$P_{\Sigma_R} = NP_{\Sigma_R}$ for $R = R_0$

A **satisfiable** conjunction $\psi(Y_1,\ldots, Y_m)$, $|code(\psi)| = m$

Transfer for $R \in$ REL

Replace: $add_d(Y_i)$ by $Y_i d$, $\quad d_j, e_j, d, e \in \{a, b\}$

$[Y_i][d_1\cdots d_k]d =/\neq [Y_j][e_1\cdots e_l]d$

by $[Y_i][d_1\cdots d_k] =/\neq [Y_j][e_1\cdots e_l]$,

Y_j by $[Y_i][d_1\cdots d_k]$ if $Y_j = [Y_i][d_1\cdots d_k]$,

Remove: $Y_i = Y_i$, $\varepsilon = \varepsilon$, $Y_i \neq Y_i d_1\cdots d_k$, $\varepsilon \neq Y_i d_1\cdots d_k$, $k\geq 1$

$[Y_i][d_1\cdots d_k]\, d \neq [Y_j][e_1\cdots e_l]\, e$ if $d \neq e$,

$\neg R([Y_i][d_1\cdots d_k]b)$, $\neg R(Y_i d_1\cdots d_{w_1} b a^{w_2})$ if $w_1 \geq w_2$

The new satisfiable system:

$Y_j = [Y_i][d_1\cdots d_k]$, $\quad Y_{i_1} \neq [Y_{i_2}][d_1\cdots d_k]$,

$[\neg]R(Y_i a^w)$, $\quad [\neg]R(Y_i d_1\cdots d_{w_1} b a^{w_2})$,

$k, w, w_1 + w_2 + 1 < m$, $\quad i, i_1, i_2 \in I \subseteq \{1,\ldots, m\}$, $j \notin I$

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Satisfiability of the new system for relations on padded strings

Satisfiability for relations on padded strings

Some Satisfiability Problems for Σ

$\mathrm{P}_{\Sigma_R} = \mathrm{NP}_{\Sigma_R}$ for $R = R_0$

⇒ $\forall s \exists r\, (R(s) \rightarrow s = (rb)_{dbl})$ & $\forall i\, \forall j\, (i \leq j-3 \rightarrow R((a^i b^{j-i})_{dbl}))$

The new satisfiable system:

$$Y_j = [Y_i][d_1 \cdots d_k], \qquad Y_{i_1} \neq [Y_{i_2}][d_1 \cdots d_k],$$

$$[\neg]R(Y_i a^w), \qquad [\neg]R(Y_i d_1 \cdots d_{w_1} b a^{w_2})$$

$k, w, w_1 + w_2 + 1 < m, \quad i, i_1, i_2 \in I \subseteq \{1, \ldots, m\}, \quad j \notin I$

Solutions:
(for $R \in$ REL)

$$R(Y_i d_1 \cdots d_{w_1} b a^{w_2}) \Leftrightarrow |Y_i| = w_2 - w_1 - 1 \ (< m),$$

$$Y_i = a^i b^{2m+1-i} a^{2m+1-w} \Rightarrow R(Y_i a^w),$$

$$Y_i = a^i b^{2m-i} a^{2m+1} \Rightarrow \neg R(Y_i a^w).$$

⇒ $i_1 \neq i_2 \Rightarrow Y_{i_1} \neq Y_{i_2}$

⇒ small strings Y_i ⇒ small strings Y_j

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Some Satisfiability Problems for Σ

Satisfiability for relations on padded strings

Some Satisfiability Problems for Σ

$P_{\Sigma_R} = NP_{\Sigma_R}$ for $R = R_0$

$\mathrm{SAT}_\Sigma \quad = \{(\boldsymbol{x}, \mathit{code}(\psi)) \in A^\infty \mid \boldsymbol{x} \in A^n \ \&\ \psi \in L_\Sigma \ \&\ \Sigma \models (\exists \boldsymbol{Y} \in A^m)\psi(\boldsymbol{x},\boldsymbol{Y})\}$

$\mathrm{SUB\text{-}SAT}_\Sigma \quad = \{(\boldsymbol{x}, \mathit{code}(\psi)) \in \mathrm{SAT}_\Sigma \mid \boldsymbol{x} \in (A^{(<k)} \cup S_{k,k+l})^n\}$

k and *l* are polynomially dependent on *n* and *m* ⇒ small strings !!!

k' and *l'* are polynomially dependent on $|x_1|,\ldots,|x_n|$ and *m* ⇒ small strings !!!

$\mathrm{RES\text{-}SAT}_\Sigma \quad = \{(\boldsymbol{x}, \mathit{code}(\psi)) \in \mathrm{SAT}_\Sigma \mid \Sigma \models (\exists \boldsymbol{Y} \in (A^{(<k')} \cup S_{k',k'+l'})^m)\ \psi(\boldsymbol{x},\boldsymbol{Y})\}$

$\mathrm{RES\text{-}SAT}_\Sigma^{(1)} \quad = \{\langle \boldsymbol{x}, \mathit{code}(\psi)\rangle_{dbl} \in A \mid (\boldsymbol{x}, \mathit{code}(\psi)) \in \mathrm{RES\text{-}SAT}_\Sigma\}$

unary

the new code

$S_{j,v} =_{\mathrm{df}} \mathit{SMALL}_{j,v} =_{\mathrm{df}} \{a^i b^{j-i} s \mid 1 \le i \le j-3 \ \&\ s \in A^{(\le v)}\},\ A^{(<k)} =_{\mathrm{df}} \{s \in A \mid |s| < k\},\ldots$

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Small codes satisfying $R \in \text{REL}$

Satisfiability for relations on padded strings

Some Satisfiability Problems for Σ

$P_{\Sigma_R} = NP_{\Sigma_R}$ for $R = R_0$

$A = \{a, b\}^*$

The code for the tuple $(s_1,\ldots, s_k) \in A^k$:

1. $\langle s_1,\ldots, s_k\rangle \in \{d_1\cdots d_n b^3 \in A \mid \forall i(d_i\cdots d_{i+3} \neq b^3 a)\}$,

2. $\langle a^i b^{j-i}, code(X_1{=}X_1)\rangle =_{df} a^i b^{j-i} \in A \qquad (1 \leq i \leq j-3)$

We get many **small codes** $\langle a^i b^{j-i}, code(X_1{=}X_1)\rangle_{dbl} \in SMALL_{j,j}$ satisfying $R \in \text{REL}$.

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The recursive definition of R_0

Satisfiability for relations on padded strings

Some Satisfiability Problems for Σ

$\mathrm{P}_{\Sigma_R} = \mathrm{NP}_{\Sigma_R}$ for $R = R_0$

$CODES = \{\langle \mathbf{x}, code(\psi)\rangle_{dbl} \in A \mid \mathbf{x} \in A^{\infty} \;\&\; \psi \in L_{\Sigma_R}\}$

$S_{j,v} = SMALL_{j,v} = \{a^i b^{j-i} s \mid 1 \le i \le j-3 \;\&\; s \in A^{(\le v)}\}$

⇨ **small strings:** $\langle a^i b^{j-i}, code(X_1 = X_1)\rangle_{dbl} = (a^i b^{j-i})_{dbl} \in S_{j,j}$ where $R((a^i b^{j-i})_{dbl})$

For $R \in \mathrm{REL}$

$\mathrm{REL}_0 = \{R \in \mathrm{REL} \mid \forall s\,(s \notin CODES \to \neg R(s))\}$

$\mathrm{REL}_k = \{R \in \mathrm{REL}_{k-1} \mid \forall s\,(s = \langle \mathbf{x}, code(\psi)\rangle_{dbl} \in A^{(=k)}$
$\to (R(s) \leftrightarrow \Sigma_R \models (\exists \mathbf{Y} \in (A^{(<k)} \cup S_{k,k+1})^m)\,\psi(\mathbf{x},\mathbf{Y})))\}$

$R_0 \in_{\mathrm{def}} \bigcap_{k \ge 1} \mathrm{REL}_k$

$\Rightarrow R_0(s) \Leftrightarrow s \in \text{RES-SAT}_{\Sigma_{R_0}}{}^{(1)}$

$s_{dbl} = sa^{|s|}$, $\langle s_1,\ldots, s_k\rangle$ code of the tuple, $S \subseteq A \Rightarrow S^{(<k)} =_{\mathrm{df}} \{s \in S \mid |s| < k\},\ldots$

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$\mathbf{SAT}_{\Sigma_{R_0}} \in \mathbf{P}_{\Sigma_{R_0}}$ and $\mathbf{P}_{\Sigma_{R_0}} = \mathbf{NP}_{\Sigma_{R_0}}$

Satisfiability for relations on padded strings

Some Satisfiability Problems for Σ

$P_{\Sigma_R} = NP_{\Sigma_R}$ for $R = R_0$

$\Sigma = \Sigma_{R_0}$

$SAT_\Sigma = RES\text{-}SAT_\Sigma$ **(quantifier domains can be restricted)**

⇓ Replace parameters by suitable small parameters

$SUB\text{-}SAT_\Sigma$ **(small parameters)** $SUB\text{-}SAT_\Sigma = SUB\text{-}SAT_\Sigma \cap RES\text{-}SAT_\Sigma$

⇓ Compute the code of a tuple and double the length

$RES\text{-}SAT_\Sigma^{(1)}$ **(unary)**

⇓ Check the new string by means of R_0

Output: a / b

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Thank you for your attention!

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