Presentations of Structures in Admissible Sets

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Let $\mathfrak{M}$ be a structure of a computable signature and let $\mathbb{A}$ be an admissible set.

Definition

A presentation of $\mathfrak{M}$ in $\mathbb{A}$ is any structure $\mathcal{C}$ such that $\mathcal{C} \cong \mathfrak{M}$ and the domain of $\mathcal{C}$ is a subset of A.

We can treat (the atomic diagram of) a presentation $\mathcal{C}$ as a subset of A, using some Gödel numbering of the atomic formulas of the signature of $\mathfrak{M}$.

Definition

The problem of presentability of $\mathfrak{M}$ in $\mathbb{A}$ is the set $\mathrm{Pr}(\mathfrak{M}, \mathbb{A})$ consisting of the atomic diagrams of all possible presentations of $\mathfrak{M}$ in $\mathbb{A}$:

$$\mathrm{Pr}(\mathfrak{M}, \mathbb{A}) = \{ \mathcal{C} \mid \mathcal{C} \text{ is a presentation of } \mathfrak{M} \text{ in } \mathbb{A} \}$$

Denote by $\underline{\mathfrak{M}}$ the set $\mathrm{Pr}(\mathfrak{M}, \mathbb{HF}(\varnothing))$ of all presentations of $\mathfrak{M}$ in the least admissible set.

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Σ-operators

A mapping $F : P(A)^n \to P(A)$ $(n \in \omega)$ is called a *Σ-operator* if there is a Σ-formula $\Phi(x_0, \ldots, x_{n-1}, y)$ of the signature $\sigma_{\mathbb{A}}$ such that for all $S_0, \ldots, S_{n-1} \in P(A)$

$$F(S_0, \ldots, S_{n-1}) = \{ a \mid \exists a_0, \ldots, a_{n-1} \in A$$

$$(\bigwedge_{i<n} a_i \subseteq S_i \wedge \mathbb{A} \models \Phi(a_0, \ldots, a_{n-1}, a))\}.$$

Suppose $B, C \subseteq A$. *B is eΣ-reducible to C* $(B \leqslant_{e\Sigma} C)$ if there exists a unary Σ-operator F such that $C \in \delta_c(F)$ and $B = F(C)$.

B is TΣ-reducible to C $(B \leqslant_{T\Sigma} C)$ if there exist binary Σ-operators F_0 and F_1 such that $\langle C, A \setminus C \rangle \in \delta_c(F_0) \cap \delta_c(F_1)$ for which $B = F_0(C, A \setminus C)$ and $A \setminus B = F_1(C, A \setminus C)$.

Here $\delta_c(F)$ is the set of elements of $P(A)^n$ in which F is strongly continuous. It is easy to show that in $\mathbb{HF}(\mathfrak{M})$ any subset belongs to $\delta_c(F)$ for any Σ-operator F.

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A mapping $F : P(A)^n \to P(A)$ ($n \in \omega$) is called a Σ*-operator* if there is a Σ-formula $\Phi(x_0, \ldots, x_{n-1}, y)$ of the signature $\sigma_{\mathbb{A}}$ such that for all $S_0, \ldots, S_{n-1} \in P(A)$

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Uniform reduciblities

Suppose $\mathcal{X}, \mathcal{Y} \subseteq P(A)$. $\mathcal{X}$ *is Medvedev reducible to* $\mathcal{Y}$ ($\mathcal{X} \leqslant \mathcal{Y}$) if there exist binary Σ-operators F_0 and F_1 such that, for all $Y \in \mathcal{Y}$, $\langle Y, A \setminus Y \rangle \in \delta_c(F_0) \cap \delta_c(F_1)$ and, for some $X \in \mathcal{X}$, $X = F_0(Y, A \setminus Y)$ and $A \setminus X = F_1(Y, A \setminus Y)$.

$\mathcal{X}$ *is Dyment reducible to* $\mathcal{Y}$ ($\mathcal{X} \leqslant_e \mathcal{Y}$) if there exists a unary Σ-operator F such that, for all $Y \in \mathcal{Y}$, $Y \in \delta_c(F)$ and $F(\mathcal{Y}) \subseteq \mathcal{X}$.

$\mathcal{X}$ *is Muchnik reducible to* $\mathcal{Y}$ ($\mathcal{X} \leqslant_w \mathcal{Y}$) if, for any $Y \in \mathcal{Y}$, there exist binary Σ-operators F_0 and F_1 such that $\langle Y, A \setminus Y \rangle \in \delta_c(F_0) \cap \delta_c(F_1)$ and, for some $X \in \mathcal{X}$, $X = F_0(Y, A \setminus Y)$ and $A \setminus X = F_1(Y, A \setminus Y)$.

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Σ-definability of structures in admissible sets

Let $\mathfrak{M}$ be a structure of relational computable signature $\langle P_0^{n_0}, \ldots, P_k^{n_k}, \ldots \rangle$ and let $\mathbb{A}$ be an admissible set.

Definition

$\mathfrak{M}$ is called Σ-definable in $\mathbb{A}$ if there exists a computable sequence of Σ-formulas $\varphi(x_0, y), \psi(x_0, x_1, y), \psi^*(x_0, x_1, y)$, $\varphi_0(x_0, \ldots, x_{n_0-1}, y), \varphi_0^*(x_0, \ldots, x_{n_0-1}, y), \ldots, \varphi_k(x_0, \ldots, x_{n_k-1}, y)$, $\varphi_k^*(x_0, \ldots, x_{n_k-1}, y), \ldots$ such that, for some parameter $a \in A$, $M_0 \leftrightharpoons \varphi^{\mathbb{A}}(x_0, a) \neq \varnothing$, $\eta \leftrightharpoons \psi^{\mathbb{A}}(x_0, x_1, a) \cap M_0^2$ is a congruence on the structure $\mathfrak{M}_0 \leftrightharpoons \langle M_0, P_0^{\mathfrak{M}_0}, \ldots, P_k^{\mathfrak{M}_0}, \ldots \rangle$, where

$$P_k^{\mathfrak{M}_0} \leftrightharpoons \varphi_k^{\mathbb{A}}(x_0, \ldots, x_{n_k-1}) \cap M_0^{n_k}, \; k \in \omega,$$
$$\psi^{*\mathbb{A}}(x_0, x_1, a) \cap M_0^2 = M_0^2 \setminus \psi^{\mathbb{A}}(x_0, x_1, a),$$

$\varphi_k^{*\mathbb{A}}(x_0, \ldots, x_{n_k-1}, a) \cap M_0^{n_k} = M_0^{n_k} \setminus \varphi_k^{\mathbb{A}}(x_0, \ldots, x_{n_k-1})$ for all $k \in \omega$, and the structure $\mathfrak{M}$ is isomorphic to the quotient structure $\mathfrak{M}_0 / \eta$.

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Motivation

For arbitrary countable structures $\mathfrak{M}$ and $\mathfrak{N}$, consider the following properties:

1) $\mathfrak{M}$ is Σ-definable in $\mathbb{HF}(\mathfrak{N})$;
2) $\underline{\mathfrak{M}} \leqslant \underline{(\mathfrak{N}, \bar{n})}$ for some $\bar{n} \in N^{<\omega}$;
3) $\underline{\mathfrak{M}} \leqslant_w \underline{\mathfrak{N}}$.

It is easy to see that, for any $\mathfrak{M}$ and $\mathfrak{N}$,

$$1 \Rightarrow 2 \Rightarrow 3.$$

Theorem (Ash, Knight, Manasse, Slaman; Chisholm)

Let $\mathfrak{M}$ be a countable structure and let $P \subseteq M^n$. Then the following are equivalent:

1) *P is Σ-definable in $\mathbb{HF}(\mathfrak{M})$;*
2) *for any $\mathcal{C} \in \underline{(\mathfrak{M}, P)}$, $\quad P^{\mathcal{C}}$ is $\mathcal{C} \restriction \sigma_{\mathfrak{M}}$-c.e.*

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Classes of adjoint structures

For countable structure $\mathfrak{M}$ consider classes

$$\mathcal{K}_\Sigma(\mathfrak{M}) = \{\mathfrak{N} \mid \mathfrak{N} \text{ is } \Sigma\text{-definable in } \mathbb{HF}(\mathfrak{M})\}$$
$$\mathcal{K}_e(\mathfrak{M}) = \{\mathfrak{N} \mid \underline{\mathfrak{N}} \leqslant_e \underline{(\mathfrak{M}, \bar{m})} \text{ for some } \bar{m} \in M^{<\omega}\}$$
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For any $\mathfrak{M}$,

$$\mathcal{K}_\Sigma(\mathfrak{M}) \subseteq \mathcal{K}_e(\mathfrak{M}) \subseteq \mathcal{K}(\mathfrak{M}),$$

as well as $\mathcal{K}(\mathfrak{M}) \subseteq \mathcal{K}_w(\mathfrak{M})$, $\mathcal{K}_e(\mathfrak{M}) \subseteq \mathcal{K}_{ew}(\mathfrak{M})$, and $\mathcal{K}_{ew}(\mathfrak{M}) \subseteq \mathcal{K}_w(\mathfrak{M})$.

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Negative result

Theorem

There exist a structure $\mathfrak{M}$ and a relation $P \subseteq M$ such that $\underline{(\mathfrak{M}, P)} \equiv_e \underline{\mathfrak{M}}$, but $(\mathfrak{M}, P)$ is not Σ-definable in $\mathbb{HF}(\mathfrak{M})$.

The proof uses the result of T. Slaman and S. Wehner: there exists a structure with problem of presentability belonging to the least nonzero degree in the Medvedev lattice.

For arbitrary structures $\mathfrak{M}$ and $\mathfrak{M}'$ of the same signature and any $n \in \omega$, we denote by $\mathfrak{M} \preccurlyeq_n^{\mathrm{HF}} \mathfrak{M}'$ the fact that $\mathbb{HF}(\mathfrak{M}) \preccurlyeq_n \mathbb{HF}(\mathfrak{M}')$. It is easy to verify that, for $n < 2$, $\mathfrak{M} \preccurlyeq_n^{\mathrm{HF}} \mathfrak{M}'$ if and only if $\mathfrak{M} \preccurlyeq_n \mathfrak{M}'$. For $n = 2$, $\mathfrak{M} \preccurlyeq_2^{\mathrm{HF}} \mathfrak{M}'$ if and only if $\mathfrak{M} \leqslant \mathfrak{M}'$ and for any computable sequence $\{\varphi_{mn}(\bar{x}_m, \bar{y}_n, \bar{z}) | m, n \in \omega\}$ of quantifier-free formulas of signature $\sigma_{\mathfrak{M}}$ and any $\bar{m} \in M^{<\omega}$,

$$\mathfrak{M}' \models \bigvee_{m \in \omega} \exists \bar{x}_m \bigwedge_{n \in \omega} \forall \bar{y}_n \varphi_{mn}(\bar{x}_m, \bar{y}_n, \bar{z})$$

implies that

$$\mathfrak{M} \models \bigvee_{m \in \omega} \exists \bar{x}_m \bigwedge_{n \in \omega} \forall \bar{y}_n \varphi_{mn}(\bar{x}_m, \bar{y}_n, \bar{z}).$$

Proposition

Let structures $\mathfrak{N}$ *and* $\mathfrak{N}'$ *be such that* $\mathfrak{N} \preccurlyeq_n^{\mathbb{HF}} \mathfrak{N}'$, $1 \leqslant n \leqslant \omega$, $\mathfrak{N}'$ *is constructivizable , and let* $\mathfrak{M}$ *be* Σ*-definable in* $\mathbb{HF}(\mathfrak{N})$. *Then*
1) if $n > 1$ *then there exists a constructivizable structure* $\mathfrak{M}'$ *such that* $\mathfrak{M} \preccurlyeq_n \mathfrak{M}'$;
2) if $n = 1$ *then there exists a partial constructivizable structure* $\mathfrak{M}'$ *such that* $\mathfrak{M} \preccurlyeq_\exists \mathfrak{M}'$.

We say that $\mathfrak{M}$ has a *constructivizable n-extension* if for some constructivizable $\mathfrak{M}'$ holds $\mathfrak{M} \preccurlyeq_n^{\mathrm{HF}} \mathfrak{M}'$.

Corollary

If $\mathfrak{M}$ *has a constructivizable n-extension then, for any* $\mathfrak{N} \in \mathcal{K}_\Sigma(\mathfrak{M})$, $\mathfrak{N}$ *also has a constructivizable n-extension.*

Negative result

Theorem

If $\mathfrak{M}$ is not constructivizable but there is a constructivizable $\mathfrak{M}'$ s.t. $\mathfrak{M} \preccurlyeq_2^{\mathrm{HF}} \mathfrak{M}'$ then

$$\mathcal{K}_\Sigma(\mathfrak{M}) \subsetneq \mathcal{K}(\mathfrak{M}).$$

Example: $(\omega_1^{CK}, \leqslant) \preccurlyeq^{\mathrm{HF}} (\omega_1^{CK}(1+\eta), \leqslant)$.

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Theorem

For any countable structures $\mathfrak{M}$ and $\mathfrak{N}$ and any $R \subseteq \mathbb{HF}(\mathfrak{N})$, the following are equivalent:

1) *for any presentation $\mathcal{C}$ of $\mathfrak{M}$ in $\mathbb{HF}(\mathfrak{N})$,*
   $R \leqslant_{e\Sigma} \mathcal{C}$;
2) *R is Σ-definable in $\mathbb{HF}(\mathfrak{M}, \mathfrak{N})$.*

Definition

Let $\mathfrak{M}$ and $\mathfrak{N}$ be a countable structures. $\mathfrak{M}$ is said to have a degree (e-degree) over $\mathfrak{N}$ if there exists a least degree in the class of $T\Sigma$-degrees ($e\Sigma$-degrees) of all possible presentations of $\mathfrak{M}$ in $\mathbb{HF}(\mathfrak{N})$.

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Theorem

Let $\mathfrak{M}$ and $\mathfrak{N}$ be a countable structure. The following are equivalent:

1) *$\mathfrak{M}$ has a degree (e-degree) over $\mathfrak{N}$;*
2) *some presentation $\mathcal{C} \subseteq HF(N)$ of $\mathfrak{M}$ is Δ-definable (Σ-definable) in $\mathbb{HF}(\mathfrak{M}, \mathfrak{N})$.*

Corollary

For a countable $\mathfrak{M}$, $\mathfrak{M}$ has a degree (e-degree) iff, for some $\mathcal{C} \in \underline{\mathfrak{M}}$, $\mathcal{C}$ is Δ-definable (Σ-definable) in $\mathbb{HF}(\mathfrak{M})$.

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Proposition

If $\mathfrak{M}$ has a degree (e-degree) over $\mathfrak{N}$ and $\mathfrak{N}$ is Σ-definable in $\mathbb{HF}(\mathfrak{N}')$ then $\mathfrak{M}$ has a degree (e-degree) over $\mathfrak{N}'$.

Proposition

For any countable structure $\mathfrak{A}$ there exists a structure $\mathfrak{M}$ which has a degree but is not Σ-definable in $\mathbb{HF}(\mathfrak{A})$.

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Positive results

Theorem

If $\mathfrak{M}$ has a degree then

$$\mathcal{K}_{\Sigma}(\mathfrak{M}) = \mathcal{K}_e(\mathfrak{M}) = \mathcal{K}(\mathfrak{M}) = \mathcal{K}_w(\mathfrak{M}).$$

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Generalization

For structures $\mathfrak{M}$ and $\mathfrak{N}$ consider classes

$$\mathcal{K}_e(\mathfrak{M}, \mathfrak{N}) = \{\mathfrak{M}' \mid \mathrm{Pr}(\mathfrak{M}', \mathbb{HF}(\mathfrak{N})) \leqslant_e \mathrm{Pr}((\mathfrak{M}, \bar{m}), \mathbb{HF}(\mathfrak{N})),\ \bar{m} \in M^{<\omega}\}$$

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For any $\mathfrak{M}$ and $\mathfrak{N}$,

$$\mathcal{K}_\Sigma(\mathfrak{M}) \subseteq \mathcal{K}_e(\mathfrak{M}, \mathfrak{N}) \subseteq \mathcal{K}(\mathfrak{M}, \mathfrak{N}).$$

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Theorem

If $\mathfrak{M}$ has a degree over $\mathfrak{N}$ then

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If $\mathfrak{M}$ has an e-degree over $\mathfrak{N}$ then

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If $\mathfrak{M}$ has an e-degree over $\mathfrak{N}$ then

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Proposition

Let $\mathfrak{N}$ be either an infinite structure of empty signature, or a dense linear ordering. Then

$$\mathcal{K}_\Sigma(\mathfrak{M}) = \mathcal{K}_e(\mathfrak{M}, \mathfrak{N}) = \mathcal{K}(\mathfrak{M}, \mathfrak{N}).$$

Locally constructivizable structures

A structure $\mathfrak{M}$ is called *locally constructivizable* if $\mathrm{Th}_\exists(\mathfrak{M}, \bar{m})$ is c.e. for any $\bar{m} \in M^{<\omega}$. For structures $\mathfrak{M}$ and $\mathfrak{N}$, we denote by $\mathfrak{M} \leqslant_\exists \mathfrak{N}$ the fact that for any $\bar{m} \in M^{<\omega}$ there is $\bar{n} \in N^{<\omega}$ such that $\mathrm{Th}_\exists(\mathfrak{M}, \bar{m}) \leqslant_e \mathrm{Th}_\exists(\mathfrak{N}, \bar{n})$. In particular, if $\mathfrak{M}$ is locally constructivizable then $\mathfrak{M} \leqslant_\exists \mathfrak{N}$ for any $\mathfrak{N}$.

Proposition

If $\mathfrak{M} \in \mathcal{K}_w(\mathfrak{N})$ *then* $\mathfrak{M} \leqslant_\exists \mathfrak{N}$.

Corollary

If $\mathfrak{M}$ *is locally constructivizable then, for any* $\mathfrak{N} \in \mathcal{K}_w(\mathfrak{M})$, $\mathfrak{N}$ *is also locally constructivizable.*

If $\mathfrak{M} \leqslant_\exists \mathfrak{N}$ then $\mathfrak{M}$ has a degree (*e*-degree) over $\mathfrak{N}$ if and only if $\mathfrak{M}$ is Σ-definable in $\mathbb{HF}(\mathfrak{N})$.

Another positive result

Theorem

Let $\mathfrak{M}$ and $\mathfrak{N}$ be a countable structures and let $R \subseteq \mathbb{HF}(\mathfrak{M})$. The following are equivalent:

1) *for any presentation $\mathcal{C}$ of $(\mathbb{HF}(\mathfrak{M}), R)$ in $\mathbb{HF}(\mathfrak{N})$,*

$$\mathcal{C} \leqslant_{e\Sigma} \mathcal{C} \upharpoonright \mathfrak{M};$$

2) *R is Σ-definable in $\mathbb{HF}(\mathfrak{M}, \mathfrak{N})$.*

Some upper semilattices embeddable into Medvedev lattice

Let $\mathcal{K}$ be the set of all countable structures with computable signatures. For any $* \in \{\Sigma, e, \ , w, ew\}$ we define a relation $\leqslant_*$ on $\mathcal{K}$ as follows: $\mathfrak{M} \leqslant_* \mathfrak{N}$ if and only if $\mathcal{K}_*(\mathfrak{M}) \subseteq \mathcal{K}_*(\mathfrak{N})$. Clearly each $\leqslant_*$ is reflexive and transitive, so we can consider partial orderings $\mathcal{S}_* = (\mathcal{K}/\equiv_*, \leqslant_*)$. Let $\mathcal{M}$ denotes the Medvedev lattice, and $\mathcal{D}, \mathcal{D}_e$ denote semilattices of Turing and enumeration degrees respectively.

Theorem

Each of $\mathcal{S}_*$, $* \in \{\Sigma, e, \ , w, ew\}$, is an upper semilattice, and there are embeddings

$$\mathcal{D} \hookrightarrow \mathcal{D}_e \hookrightarrow \mathcal{S}_\Sigma \hookrightarrow \mathcal{S}_e \hookrightarrow \mathcal{S} \hookrightarrow \mathcal{M}.$$