# Scaled Dimension of Individual Strings

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# Why we use Hausdorff dimension?

Let $X$, $Y$ complexity classes,

then $X$ and $Y$ are sets in the Cantor Space ($\{0,1\}^\infty$)

$$(A \subseteq \{0,1\}^* \Leftrightarrow \chi_A \in \{0,1\}^\infty)$$

$$\text{If } \dim_{\mathrm{H}}(X) \neq \dim_{\mathrm{H}}(Y) \quad \Rightarrow \quad X \neq Y.$$

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# Lutz's characterization

An *s*-gale is a function $d : \{0,1\}^* \to [0,\infty)$ such that

$$\frac{d(w0)+d(w1)}{2^s} = d(w).$$

$$\dim_{\mathrm{H}}(X) = \inf\{s \in [0,\infty) \mid \exists\ s-\text{gale } d \text{ s.t. } X \subseteq S^{\infty}[d]\},$$

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$$\forall S \in X, \quad \limsup_{n} d(S[0\ldots n-1]) = \infty.$$

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# Resource-bounded dimension

Using resource bounds ($\Delta$) on $d$,

$$\dim_{\Delta}(X) = \inf\{s \in [0,\infty) \mid \exists\ \Delta\text{-comp. } s\text{-gale } d \text{ s.t. } X \subseteq S^{\infty}[d]\}.$$

For example, we define constructive dimension as

$$\mathrm{cdim}(X) = \inf\{s \in [0,\infty) \mid \exists \text{ constructive } s\text{-gale } d \text{ s.t. } X \subseteq S^{\infty}[d]\}.$$

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# Scaled gales

Let $g : \mathbb{N} \times [0, \infty) \to [0, \infty)$ an scale function,

an scaled $s^g$-gale is a function $d : \{0, 1\}^* \to [0, \infty)$ such that

$$\frac{d(w0) + d(w1)}{2^{g(|w|,s) - g(|w|+1,s)}} \leq d(w).$$

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Notice that if $g(n, s) = ns$, then $\dim^g_\Delta = \dim_\Delta$.

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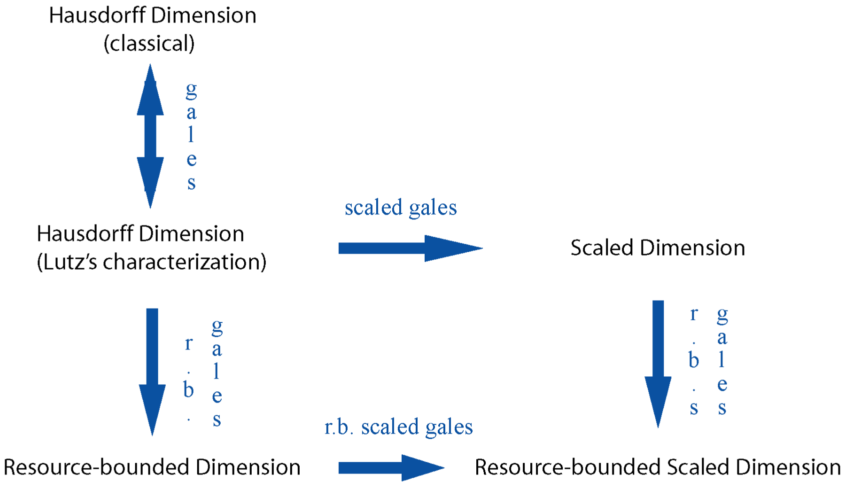
# Summary


Hausdorff Dimension (classical)
gales
Hausdorff Dimension (Lutz's characterization)
scaled gales
Scaled Dimension
r.b. gales
r.b.s gales
r.b. scaled gales
Resource-bounded Dimension
Resource-bounded Scaled Dimension

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# Definition of $\dim(w)$ (Lutz)

$\mathrm{cdim}(S) = \mathrm{cdim}(\{S\}) =$

$\inf\{s \mid \exists \text{ constructive } s\text{−gale } d \text{ s.t. } \limsup_n d(S[0 \ldots n-1]) = \infty\}.$

To define $\dim(w)$:

1. We have to replace gales by termgales.
2. We have to replace “unbounded as $n \to \infty$”.
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# Scaled Dimension of individual strings

To define scaled dimension of a finite string:

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### Theorem

*For every* $S \in \{0,1\}^\infty$,

1. *[Lutz]*
$$\mathrm{cdim}(S) = \liminf_n \dim(S[0\ldots n-1]).$$
2. *Let g be a scale function,*
$$\mathrm{cdim}^g(S) = \liminf_n \dim^g(S[0\ldots n-1]).$$

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1. *[Lutz] The Kolmogorov complexity of a string is (up an additive constant) the product of his length and its dimension.*

$$|K(w) - |w|\mathrm{dim}(w)| \leq c$$

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# Summary

- Dimensions are tools that were defined to distinguish between complexity classes.
- We can define a discrete version of constructive dimension and constructive scaled dimension.
- Constructive dimensions of (finite and infinite) sequences can be characterized in terms of Kolmogorov complexity.

# For Further Reading I

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