A Subrecursive Refinement of the Fundamental Theorem of Algebra

P. Peshev D. Skordev

Faculty of Mathematics and Computer Science
University of Sofia

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Approximation and Representation of Complex Numbers

- An *approximator* of a complex number α is any sequence $\gamma_0, \gamma_1, \gamma_2, \ldots$ of rational complex numbers such that

$$|\gamma_t - \alpha| \leq \frac{1}{t+1}, \quad t = 0, 1, 2, \ldots$$

(using $t+1$ instead of 2^t is essential for our main result).

- A *representation* of α is any 6-tuple of functions $f_1, f_2, f_3, f_4, f_5, f_6$ from $\mathbb{N}$ into $\mathbb{N}$ such that the sequence $\gamma_0, \gamma_1, \gamma_2, \ldots$ defined by

$$\gamma_t = \frac{f_1(t) - f_2(t)}{f_3(t) + 1} + \frac{f_4(t) - f_5(t)}{f_6(t) + 1} i, \quad t = 0, 1, 2, \ldots,$$

is an approximator of α.

A Recursive Refinement of FTA

- For any representations of the members of a finite sequence of complex numbers, the concatenation of these representations will be called a *representation of the sequence*.
- A proof given by Rosenbloom in 1945 shows implicitly that for any positive integer N there is a 6-tuple of recursive operators that transform any representation of a sequence of arbitrary complex numbers $\alpha_0, \alpha_1, \dots, \alpha_{N-1}$ into some representation of some root of the corresponding polynomial

$$P(z) = z^N + \alpha_{N-1} z^{N-1} + \dots + \alpha_1 z + \alpha_0$$

Our Subrecursive Refinement of FTA

- Our main result strengthens the previous refinement of FTA by replacing "recursive operators" with "computable operators from the second Grzegorczyk class".
- This result would be not valid if we used 2^t instead of $t+1$ in the definition of approximator. Suppose it is. Then, by applying it to the polynomial $P(z) = z^2 - 2$, we can conclude that $\sqrt{2}$ has a representation conforming to such a definition of approximator and consisting of functions from Grzegorczyk class $\mathcal{E}^2$. Since all functions from $\mathcal{E}^2$ are bounded by polynomials, we get a contradiction by an easy application of Liouville's approximation theorem.

Application of Liouville's Approximation Theorem

By Liouville's Theorem, a positive number c exists such that

$$\left|\frac{p}{q} - \sqrt{2}\right| > \frac{c}{q^2}$$

for any integer p and any positive integer q. If we had 2^t instead $t+1$ in the definition of representation, and $\bar{f}$ is a representation of $\sqrt{2}$, then we would have

$$\left|\frac{f_1(t) - f_2(t)}{f_3(t) + 1} - \sqrt{2}\right| \leq \frac{1}{2^t}$$

and consequently

$$\frac{c}{(f_3(t)+1)^2} < \frac{1}{2^t}$$

for all t in $\mathbb{N}$. This is impossible with a function $f_3 \in \mathcal{E}^2$.

$\mathcal{E}^2$-Computable Operators

The computable operators from the second Grzegorczyk class will be called further $\mathcal{E}^2$-*computable operators*. They can be constructed by means of the following constructions, where $\bar{f}$ is an abbreviation for an arbitrary n-tuple $f_1, \ldots, f_n$ of one-argument functions in $\mathbb{N}$, and $\bar{x}$ denotes an arbitrary k-tuple of natural numbers:

- $\Gamma(\bar{f}) = g$, for any function g of the class $\mathcal{E}^2$
- $\Gamma(\bar{f}) = f_j, \; j = 1, \ldots, n$
- $\Gamma(\bar{f})(\bar{x}) = \Gamma_0(\bar{f})(\Gamma_1(\bar{f})(\bar{x}), \ldots, \Gamma_m(\bar{f})(\bar{x}))$
- $\Gamma(\bar{f})(0, \bar{x}) = \Gamma_0(\bar{f})(\bar{x})$,
  $\Gamma(\bar{f})(t+1, \bar{x}) = \min\{\Gamma_1(\bar{f})(\Gamma(\bar{f})(t, \bar{x}), t, \bar{x}), \Gamma_2(\bar{f})(t, \bar{x})\}$

$\mathcal{E}^2$-Computable Functions in $\mathbb{C}$

A function $\varphi : \mathbb{C}^m \to \mathbb{C}$ will be called $\mathcal{E}^2$-*computable* if $\mathcal{E}^2$-computable operators $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5, \Gamma_6$ exist such that, whenever $\bar{f}$ is a representation of some N-tuple $\zeta_1, \ldots, \zeta_m$ of complex numbers, then the corresponding 6-tuple $\Gamma_1(\bar{f}), \Gamma_2(\bar{f}), \Gamma_3(\bar{f}), \Gamma_4(\bar{f}), \Gamma_5(\bar{f}), \Gamma_6(\bar{f})$ is a representation of the complex number $\varphi(\zeta_1, \ldots, \zeta_m)$.

Example

For any fixed $N \in \mathbb{N}$ the value of $z^N + \alpha_{N-1} z^{N-1} + \cdots + \alpha_1 z + \alpha_0$ and its modulus are $\mathcal{E}^2$-computable functions of $\alpha_0, \ldots, \alpha_{N-1}, z$.

$\mathcal{E}^2$-Computable Functions are not enough for FTA

Remark. Except for the case of $N = 1$, there is no $\mathcal{E}^2$-computable function $\varphi : \mathbb{C}^N \to \mathbb{C}$ such that for any complex numbers $\alpha_0, \alpha_1, \ldots, \alpha_{N-1}$ and $z = \varphi(\alpha_0, \alpha_1, \ldots, \alpha_{N-1})$ the equality

$$z^N + \alpha_{N-1} z^{N-1} + \cdots + \alpha_1 z + \alpha_0 = 0$$

holds. This follows from the non-existence of a continuous function with such a property (the existence of such a continuous function would imply the existence of a continuous function transforming each complex number into some of the N-th roots of it!).

The Basic Technical Result

Theorem (Rosenbloom's Basic Technical Result)

Let $P(z) = z^N + \alpha_{N-1}z^{N-1} + \cdots + \alpha_1 z + \alpha_0$, *where* $N \in \mathbb{N}^+$, $\alpha_0, \alpha_1, \ldots, \alpha_{N-1} \in \mathbb{C}$. *Let*

$$A = \max\{|\alpha_0|, |\alpha_1|, \ldots, |\alpha_{N-1}|, 1\}, \quad \gamma = \binom{N+1}{[(N+1)/2]},$$

$a \geq 5NA$, $K = 2^{(3N/2)+6}\gamma^3 A^3 a^{3N+3}$. *If* $\varepsilon > 0$. $n \in \mathbb{N}$, $n > K/\varepsilon^3$, *then*

$$\left| P\left(\frac{(u+vi)a}{n}\right) \right| < \varepsilon$$

for some integers u *and* v *with* $|u| \leq n$, $|v| \leq n$.

The Main Lemma in Rosenbloom's Proof

Lemma (Rosenbloom's Main Lemma)

Let $P(z) = z^N + \alpha_{N-1}z^{N-1} + \cdots + \alpha_1 z + \alpha_0$*, where* $N \in \mathbb{N}^+$*,* $\alpha_0, \alpha_1, \ldots, \alpha_{N-1} \in \mathbb{C}$*. Let* $0 < \varepsilon < 1$*. Then we can find complex numbers* $z_1, \ldots, z_N$ *such that*

$$|P(z_j)| < \varepsilon, \quad j = 1, \ldots, N,$$

and such that if $|P(z)| < \delta$*, where* $\varepsilon \leq \delta < 1$*, then*

$$\min_{1 \leq j \leq N} |z_j - z| < 2\delta^{1/2^N}.$$

We strengthen this lemma by adding the word "rational" after the word "find" and by replacing $2\delta^{1/2^N}$ with $2\delta^{1/2^{N-1}}$.

Concluding Part of Rosenbloom's Proof

The concluding part of Rosenbloom's proof is in his Theorem 3, where his Main Lemma is used to construct a sequence $z_1, z_2, \ldots$ of complex numbers that converges to a root of the polynomial. The lemma is applied with $\varepsilon = 2^{-n2^N}$, $n = 1, 2, \ldots$, and the inequality $|z_{n+1} - z_n| < 2^{1-n}$ for the members of the constructed sequence $z_1, z_2, \ldots$ holds. Our strengthening of the Main Lemma allows even to choose rational complex numbers $z_1, z_2, \ldots$ Unfortunately the exponential dependence of 2^{-n2^N} on n is an obstacle to realize such a construction of the sequence by means of $\mathcal{E}^2$-computable operators. Therefore we change the construction by applying the main lemma with ε of the form $2^{-2^{N-1}}(n+1)^{-2^N}$, $n = 1, 2, \ldots$. This leads to the inequality $|z_{n+1} - z_n| < (n+1)^{-2}$ that still gives an admissible rate of convergence.

Construction of the Needed Operators – First Theorem

Theorem (Corresponding to Rosenbloom's Basic Technical Result)

For any positive integer N *there are* $\mathcal{E}^2$*-computable operators* $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5, \Gamma_6$ *such that, whenever* $\bar{f}$ *is a representation of an* N*-tuple* $\alpha_0, \alpha_1, \ldots, \alpha_{N-1}$ *of complex numbers, and* $P(z)$ *is the polynomial corresponding to this* N*-tuple, then*

$$\left| P\left(\frac{\Gamma_1(\bar{f})(t) - \Gamma_2(\bar{f})(t)}{\Gamma_3(\bar{f})(t) + 1} + \frac{\Gamma_4(\bar{f})(t) - \Gamma_5(\bar{f})(t)}{\Gamma_6(\bar{f})(t) + 1} i \right) \right| < \frac{1}{t+1}$$

for all $t \in \mathbb{N}$.

The theorem proof makes use of the statement of Rosenbloom's Basic Technical Result without using details from its proof.

Construction of the Needed Operators – Second Theorem

Theorem (Corresponding to Rosenbloom's Main Lemma)

For any positive integer N there are $\mathcal{E}^2$-computable operators Γ_{kj}, $k = 1, 2, 3, 4, 5, 6$, $j = 1, 2, \ldots, N$, such that, whenever $\bar{f}$ is a representation of an N-tuple $\alpha_0, \alpha_1, \ldots, \alpha_{N-1}$ of complex numbers, and $P(z)$ is the corresponding polynomial, then for any $t \in \mathbb{N}$ and

$$z_j = \frac{\Gamma_{1j}(\bar{f})(t) - \Gamma_{2j}(\bar{f})(t)}{\Gamma_{3j}(\bar{f})(t) + 1} + \frac{\Gamma_{4j}(\bar{f})(t) - \Gamma_{5j}(\bar{f})(t)}{\Gamma_{6j}(\bar{f})(t) + 1} i\,,\ j = 1, 2, \ldots, N,$$

the inequalities $|P(z_j)| < (t+1)^{-1}$, $j = 1, 2, \ldots, N$, hold, and

$$\min_{1 \le j \le N} |z_j - z| < 2\delta^{1/2^{N-1}}$$

for all δ and z satisfying $(t+1)^{-1} \le \delta < 1$, $|P(z)| < \delta$.

About the Proof of the Second Theorem

The proof of the Second Theorem is an operator refinement of the one of our strengthening of Rosenbloom's Main Lemma. In the case of $N = 1$ we, roughly speaking, use rational approximations of the number $-\alpha_0$ that can be constructed by means of the representation $\bar{f}$ of α_0. For the inductive step, we suppose the existence of the needed $6(N-1)$-tuple of $\mathcal{E}^2$-computable operators for the case of polynomials of degree $N-1$, and use them to construct the needed $6N$-tuple of ones for polynomials of degree N, making use also of the $\mathcal{E}^2$-computable operators from Theorem 1 for this case.

Construction of the Needed Operators for the Main Result

Theorem (Our Main Result)

For any positive integer N there are $\mathcal{E}^2$-computable operators $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5, \Gamma_6$ such that, whenever $\bar{f}$ is a representation of an N-tuple $\alpha_0, \alpha_1, \ldots, \alpha_{N-1}$ of complex numbers, and $P(z)$ is the corresponding polynomial, then the 6-tuple of the functions $\Gamma_1(\bar{f}), \Gamma_2(\bar{f}), \Gamma_3(\bar{f}), \Gamma_4(\bar{f}), \Gamma_5(\bar{f}), \Gamma_6(\bar{f})$ is a representation of some root of $P(z)$.

Of course the operators Γ_{kj}, $k = 1, 2, 3, 4, 5, 6$, $j = 1, 2, \ldots, N$, from the Second Theorem are used in the proof.

Decomposition into linear factors

Corollary

For any positive integer N *there are* $\mathcal{E}^2$*-computable operators* Γ_{kj}, $k = 1, 2, 3, 4, 5, 6$, $j = 1, 2, \ldots, N$, *such that, whenever* $\bar{f}$ *is a representation of an* N*-tuple of complex numbers* $\alpha_0, \alpha_1, \ldots, \alpha_{N-1}$, *and* $P(z)$ *is the corresponding polynomial, then the 6-tuples*

$$\Gamma_{1j}(\bar{f}), \Gamma_{2j}(\bar{f}), \Gamma_{3j}(\bar{f}), \Gamma_{4j}(\bar{f}), \Gamma_{5j}(\bar{f}), \Gamma_{6j}(\bar{f}), \; j = 1, 2, \ldots, N,$$

are representations of some complex numbers $z_1, z_2, \ldots, z_N$ *with the property that for all* z *the equality*

$$P(z) = (z - z_1)(z - z_2) \cdots (z - z_N)$$

holds.

Closedness of Grzegorczyk Classes

Corollary

If an N-tuple of complex numbers $\alpha_0, \alpha_1, \ldots, \alpha_{N-1}$ *has a representation consisting of functions from Grzegorczyk class* $\mathcal{E}^m$*, where* $m \geq 2$*, then any root of the corresponding polynomial has a representation consisting of functions from the same class* $\mathcal{E}^m$*.*

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