Coinductive Proofs for Basic Real Computation

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Motivation

- Prove properties of signed digit representation of real numbers using coinductive methods

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- Prove properties of signed digit representation of real numbers using coinductive methods

- Why use coinduction?:
  - logically simple
  - easy to implement

Representing Real Numbers by Signed Digit Streams

$SD = \{-1, 0, 1\}$	the set of signed digits
$ds = (d_0 : d_1 : d_2 : \dots)$	a signed digit stream

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- $$SDToReal(ds, k) = 2^k \cdot \sum_{n=0}^{\infty} d_n \cdot 2^{-(n+1)}$$

- Coinductive definition of $(ds, k) \sim r$: The largest relation $\sim$ satisfying

$$(ds, k) \sim r \Rightarrow |r| \leq 2^k$$
$$\wedge (tail(ds), k - 1) \sim r - 2^{k-1} \cdot head(ds)$$

Closure Properties of Coinductive Relations

Given $f : X \to X$ and $B \subseteq X$, let r be coinductively defined by $r(x) \Rightarrow B(x) \wedge r(f(x))$.

<u>Lemma 1</u>
Given $h : X \to X$, s.t. for all $x \in X$, if $r(x)$ then

- $f(h(x)) = h(g(x))$
- $B(h(x))$

Then $\forall x \in X \Big(r(x) \Rightarrow r(h(x)) \Big)$.

In our case, $X = [SD] \times \mathbf{Z} \times \mathbf{R}$, $r = \sim$, $h \simeq average$.

Correctness of Coinductively Defined Signed Digit representation

Theorem

For all $k \in \mathbf{Z}, r \in \mathbf{R}$, $(ds, k) \sim r \Leftrightarrow SDToReal(ds, k) = r$.

Lemma 2: For every $x \in [-1, 1]$**,**
$ds \sim' x \Leftrightarrow SDToReal(ds, 0) = x$.
Lemma 3: For every $k \in \mathbf{Z}, r \in \mathbf{R}$**,**
$(ds, k) \sim r \Rightarrow ds \sim' 2^{-k} \cdot r$.
Lemma 4: For every $k \in \mathbf{Z}, x \in [-1, 1]$**,**

$ds \sim' x \Rightarrow (ds, k) \sim 2^k \cdot x$.

Correctness of Coinductively Defined Signed Digit representation

Lemma 2

For every $x \in [-1, 1]$, $ds \sim' x \Leftrightarrow SDToReal(ds, 0) = x$.

Proof sketch:

$'' \Longrightarrow ''$ **That is to show**

$$\forall n \in \mathbf{N} \quad ds \sim' x \Rightarrow |SDToReal(ds, 0) - x| \leq 2^{1-n}$$

By induction on n.

$'' \Longleftarrow ''$ **By coinduction principle.**

Lemma 3 and Lemma 4 can be proved by Lemma 1.

Coinductive Representation by Cauchy Sequences

- Coinductive definition of $(xs, l) \sim^c r$:

$$(xs, l) \sim^c r \Rightarrow$$
$$|head(xs) - r| \leq 2^l \wedge (tail(xs), l-1) \sim^c r$$

Coinductive Representation by Cauchy Sequences

- Coinductive definition of $(xs, l) \sim^c r$:

$$(xs, l) \sim^c r \Rightarrow$$
$$\qquad |head(xs) - r| \leq 2^l \wedge (tail(xs), l-1) \sim^c r$$

- Lemma 5: For every $(xs, l) \in [\mathbf{Q}] \times \mathbf{Z}, r \in \mathbf{R}$, $(xs, l) \sim^c r \Leftrightarrow \forall n. |xs_n - r| \leq 2^{l-n}$

Adequacy of Signed Digit Stream Representation

We want: computable functions

$$SDTC : [SD] \times \mathbf{Z} \to [\mathbf{Q}] \times \mathbf{Z}$$
$$CTSD : [\mathbf{Q}] \times \mathbf{Z} \to [SD] \times \mathbf{Z}$$

such that for all $r \in \mathbf{R}$,

1. $\forall (ds, k) \in [SD] \times \mathbf{Z} \quad \big((ds, k) \sim r \Rightarrow SDTC(ds, k) \sim^c r\big)$
2. $\forall (xs, l) \in [\mathbf{Q}] \times \mathbf{Z} \quad \big((xs, l) \sim^c r \Rightarrow CTSD(xs, l) \sim r\big)$

Adequacy of Signed Digit Stream Representation

Then for every $k \in \mathbf{N}$, we define

$SDTC(ds, k) := \big(SDTC'(k, 0, ds), k + 1\big)$ where

$$SDTC' : \mathbf{N} \times \mathbf{Q} \times [SD] \to [\mathbf{Q}]$$

$$SDTC'(k, q, ds) = \big(q + 2^{k-1} \cdot head(ds)\big)$$

$$: SDTC'(k - 1, q + 2^{k-1} \cdot head(ds), tail(ds)).$$

Adequacy of Signed Digit Stream Representation

$$
\begin{aligned}
CTSD \quad & (xs, l) := CTSD'(0, (xs, l)) \quad where \\
CTSD' \quad & : \mathbf{N} \times [\mathbf{Q}] \times \mathbf{Z} \to [SD] \times \mathbf{Z} \\
CTSD' \quad & (n, (xs, l)) = (d_0 : ds, k + 1) \quad where \\
k \quad & = \max(l, 1 + log_2|y|) \\
y \quad & = head(xs) - n \\
d_0 \quad & = \begin{cases} 0 & \text{if } |y| \leq 2^{k-1} \\ -1 & \text{if } y < -2^{k-1} \\ 1 & \text{if } y > 2^{k-1} \end{cases} \\
ds \quad & = fst\big(CTSD'((n + 2^{k-1} \cdot head(ds)), \\
& \quad (tail(xs), l - 1))\big)
\end{aligned}
$$

Average of Signed Digit Streams

Definition of function $carry$

$$carry(a, b) = \begin{cases} 1 & \text{if } a_0 + b_0 = 2 \\ 0 & \text{if } a_0 + b_0 = 0 \\ -1 & \text{if } a_0 + b_0 = -2 \\ 1 & \text{if } a_0 + b_0 = 1 \wedge a_1 + b_1 > 0 \\ 0 & \text{if } a_0 + b_0 = 1 \wedge a_1 + b_1 \leq 0 \\ -1 & \text{if } a_0 + b_0 = -1 \wedge a_1 + b_1 < 0 \\ 0 & \text{if } a_0 + b_0 = -1 \wedge a_1 + b_1 \geq 0 \end{cases}$$

Average of Signed Digit Streams

- For every $a, b \in [SD]$, we define the auxiliary function avA as follows.

$$avA : [SD] \to [SD] \to [SD]$$
$$avA(a,b) = (head(a) + head(b) - 2 \cdot carry(a,b) + carry(tail(a), tail(b)))$$
$$: avA(tail(a), tail(b))$$

Average of Signed Digit Streams

- For every $a, b \in [SD]$, we define the auxiliary function avA as follows.

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\begin{aligned}
&avA : [SD] \rightarrow [SD] \rightarrow [SD] \\
&avA(a,b) = (head(a) + head(b) - \\
&\qquad 2 \cdot carry(a,b) + carry(tail(a), tail(b))) \\
&\qquad : avA(tail(a), tail(b))
\end{aligned}
$$

- For every $a, b \in [SD]$, we define average function $av : [SD] \rightarrow [SD] \rightarrow [SD]$
  $av(a,b) = carry(a,b) : avA(a,b)$

Average of Signed Digit Streams

- Lemma 7: For every $a, b \in [SD], x, y \in [-1, 1]$,

$$a \sim' x \wedge b \sim' y \Rightarrow avA(a,b) \sim' x + y - carry(a,b)$$

Average of Signed Digit Streams

- Lemma 7: For every $a, b \in [SD], x, y \in [-1, 1]$,

$$a \sim' x \wedge b \sim' y \Rightarrow avA(a, b) \sim' x + y - carry(a, b)$$

- Lemma 8: For every $a, b \in [SD]$, $x, y \in [-1, 1]$,

$$a \sim' x \wedge b \sim' y \Rightarrow av(a, b) \sim' (x + y)/2$$

Minlog

The proof for the average function based on signed digit streams has been developed in Minlog.

Minlog, developed by the Logic Group of the Department of Mathematics at the University of Munich, is an interactive prover based on minimal logic.

Summary

- We describe two representations for real numbers, signed digit streams and Cauchy sequences.

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- We describe two representations for real numbers, signed digit streams and Cauchy sequences.
- Using coinduction, we have proved the correctness of basic arithmetic operations and ones that convert between signed digit streams and Cauchy sequences.

Future Work

- to perform coinductive proofs of multiplication and division functions for the signed digit stream representation

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- to perform coinductive proofs of multiplication and division functions for the signed digit stream representation
- to use coinductive methods to prove the correctness of algorithms based on the LFT method

Thank you!