Fibres

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... and back to category theory

Example

Let $\Sigma = (S, TF)$ be a signature and Φ be a set of FOL formulae:

1. **SPres** is the category of strict presentation with:
   - objects: $< \Sigma, \Phi >$,
   - morphism $\sigma :< \Sigma, \Phi > \rightarrow < \Sigma', \Phi' >$ is a sig. morph. $\sigma : \Sigma \rightarrow \Sigma'$ such that $\sigma(\Phi) \subseteq \Phi'$.
2. **Pres** is the category of presentation with:
   - objects: $< \Sigma, \Phi >$,
   - morphism $\sigma :< \Sigma, \Phi > \rightarrow < \Sigma', \Phi' >$ is a sig. morph. $\sigma : \Sigma \rightarrow \Sigma'$ such that $\sigma(c_\Sigma(\Phi)) \subseteq c_{\Sigma'}(\Phi')$.
3. **Theo** is the category of theories with:
   - objects: $< \Sigma, \Phi >$, where $\Phi = c_\Sigma(\Phi)$.
   - morphism $\sigma :< \Sigma, \Phi > \rightarrow < \Sigma', \Phi' >$ is a sig. morph. $\sigma : \Sigma \rightarrow \Sigma'$ such that $\sigma(\Phi) \subseteq \Phi'$.

Closure of a set of formulas

Definition

Let L be an algebra logic [*Loeckx et al*], Σ a signature, $\phi \subseteq L(\Sigma)$ a set of formulas. The closure of ϕ is the set of formulae:

$$\phi^* = \{\varphi \in L(\Sigma) \mid \phi \models \varphi\}$$

Definition (Logical consequence)

Let L be an algebra logic, Σ a signature, $\phi \in L(\Sigma)$ a formula , $\Phi \subseteq L(\Sigma)$ a set of formulas and $\mathcal{U}$ be a Σ-domain.

- ϕ is called *logical consequence* of Φ in $\mathcal{U}$, if $A \models_\Sigma \phi$, for each $A \in Mod_{\mathcal{U},\Sigma}(\Phi)$; one writes $\Phi \models_{\mathcal{U},\Sigma} \phi$.

Monoid specification in CASL

```
spec  CommMonoid1 =
          sort Elem
          ops  n: Elem;
                  _*_ : Elem x Elem -> Elem
          vars x,y,z: elem
               . n * x = x
               . (x * y) * z = x * (y * z)
               . x * y = y * x
end

spec  CommMonoid2 =
          sort Elem
          ops  n: Elem;
                  _*_ : Elem x Elem -> Elem
          vars x,y,z: elem
               . x * n = x
               . (x * y) * z = x * (y * z)
               . x * y = y * x
end
```

Concrete category

Definition

Let X be a category.
A **concrete category** over X is a pair $< D, \upsilon >$, where $\upsilon : D \rightarrow X$ is a faithful functor.

Concrete categories over **SET** are called **constructs**.

X is sometimes called **base category** of $< D, \upsilon >$.

Example: concrete category

Examples

Theo, **Pres** and **Spres**

are concrete category over the category **Sign**.

- *sign* : **SPres** $\longmapsto$ **Sign**

  $$sign : < \Sigma, \Phi > \rightarrow < \Sigma >$$

- *sign* : **Pres** $\rightarrow$ **Sign**
- *sign* : **Theo** $\rightarrow$ **Sign**

where **Sign** is the category of signatures with:

- objects= $< \Sigma >$, and
- morphism= $\sigma : \Sigma \rightarrow \Sigma'$ are signature morphism.

Fibres

Definition

Given a *concrete category* $< D, \upsilon >$ over C and a C-object c.

- The **fibre** of c is the preordered class consisting of objects d of D with $\upsilon(d) = c$, ordered by

  $$d_1 \leq d_2 \text{ iff } id_c : \upsilon(d_1) \rightarrow \upsilon(d_2) \text{ is a } D\text{-morphism.}$$

Definition

A *concrete category* $< D, \upsilon >$ over C is called:

- **Amnestic** provided its fibres are partially ordered:

  $$d_1 \leq_c d_2 \text{ and } d_2 \leq_c d_1 \text{ implies } d_1 = d_2$$

  for all C-objects c and objects d_1, d_2 in the fibre of c.
- **Fibre-complete** if its fibres are complete lattices.
- **Fibre-discrete** if its fibres are ordered by equality.

Examples

- **SPres** and **Theo** are amnestic .
- **Pres** is not amnestic .
- Fibre-discrete categories they are such that the extension that **D** makes over the objects of **C** is inessential, i.e. it has no intrinsic structure or meaning.

Concrete functors

Definition

A **concrete functor** φ between two concrete categories $< D_1, \upsilon_1 >$ and $< D_2, \upsilon_2 >$ over the same underlying category C is a functor $\varphi : D_1 \rightarrow D_2$ such that $\upsilon_1 = \varphi; \upsilon_2$.

Examples

- $\varphi :< \mathbf{Set}, \mathbf{id_{Set}} > \rightarrow < \mathbf{Set}, \mathbf{id_{Set}} >$ is a *concrete functor*.
- $\varphi : \mathbf{Rng} \rightarrow \mathbf{Ab}$ that 'forgets' multiplication is a *concrete functor*.

Proposition

1. *Every concrete functor is faithful.*
2. *Given* φ *and* ψ *between two concrete categories* $< D_1, v_1 >$ *and* $< D_2, v_2 >$, $\varphi = \psi$ *if, for every* D_1*-object* d, $\varphi(d) = \psi(d)$.

Concrete Subcategories

Definition

Let $< D, v >$ be a concrete category over X and A is a subcategory of D with inclusion $i : A \hookrightarrow D$, then

$$< A, v; i > \text{ is a } \textbf{concrete subcategory} \text{ of } < D, v >.$$

Generalised definition of **fibres**

Definition

Consider a functor $\varphi : D \rightarrow C$

- Given a C-object c, the **fibre of** c is the subcategory of D that consists of all the objects d that are mapped to c, such that $\varphi(d) = c$, together with D-morphisms $f : d_1 \rightarrow d_2$ such that $\varphi(f) = id_c$
- The functor φ is said to be **amnestic** if, in its fibres, no two distinct objects are isomorphic. That is :
  - given an isomorphism $f : d_1 \rightarrow d_2$ such that $\varphi(f) = id_c$ for some object c of C,

  then f is itself an identity.

D(c) : **fibre of c**

(Co)Cartesian morphisms

Definition

Let $\varphi : D \to C$ be a functor and $f : c \to c'$ a C-morphism.

1. Let $d : \mathbf{D}(c)$, a D-morphism $g : d \to d'$ is **co-cartesian** of f and d iff:
   - $\varphi(g) = f$
   - $\forall g' : d \to d''$ and $f' : c' \to \varphi(d'')$ such that $\varphi(g') = f; f'$, there is a *unique morphism* $h : d' \to d''$ such that $\varphi(h) = f'$ and $g' = g; h$
2. Let $d' : \mathbf{D}(c')$, a D-morphism $g : d \to d'$ is **cartesian** of f and d' iff:
   - $\varphi(g) = f$
   - $\forall g' : d'' \to d'$ and $f' : \varphi(d'') \to c$ such that $\varphi(g') = f'; f$, there is a *unique morphism* $h : d'' \to d$ such that $\varphi(h) = f'$ and $g' = g; h$

(Co)Fibrations

Definition

Let $\varphi : D \to C$ be a functor

- φ is a **fibration** if, for every C-morphism $f : c \to c'$ and D-object d' in the fibre of c', there is a *cartesian morphism* for f and d'.
- φ is a **cofibration** if, for every C-morphism $f : c \to c'$ and D-object d in the fibre of c, there is a *co-cartesian morphism* for f and d'.

Specification as (Co)Fibrations

Example

Given a signature morphism $f : \Sigma \rightarrow \Sigma'$.

1. In **SPres**:
   - $f :< \Sigma, f^{-1}(\Phi') > \rightarrow < \Sigma', \Phi' >$ is a cartesian morphism for $< \Sigma', \Phi' >$.
   - $f :< \Sigma, \Phi > \rightarrow < \Sigma', f(\Phi') >$ is a co-cartesian morphism for $< \Sigma', \Phi' >$.
2. In **Pres**:
   - $f :< \Sigma, f^{-1}(c(\Phi')) > \rightarrow < \Sigma', \Phi' >$ is a cartesian morphism for $< \Sigma', \Phi' >$.
   - $f :< \Sigma, \Phi > \rightarrow < \Sigma', f(\Phi') >$ is a co-cartesian morphism for $< \Sigma', \Phi' >$.
3. In **Theo**:
   - $f :< \Sigma, f^{-1}(\Phi') > \rightarrow < \Sigma', \Phi' >$ is a cartesian morphism for $< \Sigma', \Phi' >$.
   - $f :< \Sigma, \Phi > \rightarrow < \Sigma', c(f(\Phi')) >$ is a co-cartesian morphism for $< \Sigma', \Phi' >$.

Cleavages, cloven fibrations

Definition

Let $\varphi : D \rightarrow C$ be a functor.
A *choice* of a *cartesian morphism* for every C-morphism $f : c \rightarrow \varphi(d')$ and D-object d' is called a **cleavage**.

A fibration equipped with a cleavage is called **cloven**.

Proposition

Let $\phi : D \to C$ be a cloven fibration and $f : c \to c'$ a C-morphism.

1. *The morphism f defines a functor $f^{-1} : D(c') \to D(c)$ as follows:*
   - *Given $d : D(c'), f^{-1}(d')$ is the source of the Cartesian morphism $\phi_{f,d'} : d \to d'$ that the cleavage associates with the fibration.*
   - *Given $g : d_1 \to d_2$ in $D(c'), f^{-1}(g)$ is the morphism $f^{-1}(d_1) \to f^{-1}(d_2)$ that results from the universal property of the Cartesian morphism $\phi_{f,d_2} : f^{-1}(d_2) \to d_2$ when applied to $\phi_{f,d_1}; g$ and id_c.*
2. *The morphism f defines a functor $f : D(c) \to D(c')$ in the dual way, i.e. by working on the target side of the co-Cartesian morphism.*

What if $f = id_c$ or $f = f_1; f_2$?

Proposition

Let $\phi : D \to C$ be a functor.

1. *Given a C-object c and an object d in the fibre of c, the identity id_d is both Cartesian and co-Cartesian morphism for id_c and d.*
2. *Given C-morphisms $f_1 : c_1 \to c_2$ and $f_2 : c_2 \to c_3$,*
   - *an object d in the fibre of c_1, and*
   - *co-Cartesian morphisms $g_1 : d \to f_1(d)$ and $g_2 : f_1(d) \to f_2(f_1(d))$,*

   the composition $g_1; g_2$ provides a co-Cartesian morphism for $f_1; f_2$ and d.

Next week...

- Fibre completness
- Grothendieck Construction

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