Fibres II

Temesghen Kahsai

21/02/ 2007

Table of Contents

Last week

1. Fibres in Concrete categories
   - Concrete category
   - Fibres, Amnestic concrete category ...
   - Concrete functors
2. Generalization of Fibres to arbitrary functors
   - (Co) Cartesian Morphism
   - (Co) Fibrations
   - Example: Specification as (Co) Fibrations

Today

1. Review:
   - (Co) Cartesian Morphism
   - (Co) Fibrations
   - Example: Specification as (Co) Fibrations
2. Category of closure systems & topological spaces
3. Cloven Fibration
   - Functor between fibres
4. Split Fibration
5. Fibre Completeness
   - Lifting limits

Concrete category

Definition

Let X be a category.
A **concrete category** over X is a pair $< D, \upsilon >$, where $\upsilon : D \rightarrow X$ is a faithful functor.

Concrete categories over **SET** are called **constructs**.

X is sometimes called **base category** of $< D, \upsilon >$.

Example: Closure Systems

A closure system is a pair $< L, c >$ where L is a set and $c : 2^L \to 2^L$ is a total function satisfying the following properties:

- *Reflexivity*: for every $\Phi \subseteq L,\ \Phi \subseteq c(\Phi)$.
- *Monotonicity*: for every $\Phi, \Gamma \subseteq L,\ \Phi \subseteq \Gamma$ implies $c(\Phi) \subseteq c(\Gamma)$.
- *Idempotence*: for every $\Phi \subseteq L,\ c(c(\Phi)) \subseteq c(\Phi)$.

Definition

We define the category of *closure systems* **CLOS** whose morphisms

$$f :< L, c > \to < L', c' >$$

are the maps $f : L \to L'$ such that $f(c(\Phi)) \subseteq c'(f(\Phi))$ for all $\Phi \subseteq L$.

(**CLOS**, υ) is *concrete* over **Set**.

Topological space

A topological space is a pair (T, C) consisting of a set T and a family C of subsets of T, the so-called *closed sets* , satisfying the following axioms:

1. $\emptyset, T \in C$
2. $\bigcap_{i \in I} S_i \in C$ for $S_i \in C, i \in I$.
3. $(S_1 \cup S_2) \in C$ for $S_i \in C, i \in \{1, 2\}$.

Definition

Category of *topological spaces* **Top** whose:

- *objects*: <T,C> and
- *morphisms*: $f :< T, C > \rightarrow < T', C' >$ are those maps $f : T \rightarrow T'$ such that $f^{-1}(c') \in C$, for $c' \in C'$.

$(\mathbf{Top}, \upsilon)$ is concrete over **Set**.

Top & CLOS

Topological spaces are closure systems with addition of the following axiom:

- $c(A \cup B) = c(A) \cup c(B) \qquad *$

Given a *topological space* $(T, C) \rightsquigarrow$ construct the equivalent closure system by defining:

$$C(x) := \bigcap_{S \in C, x \in S} S$$

Given a *closure system* (+ *) $\rightsquigarrow$ construct a topological space (T, C) by defining:

$$x \subseteq T \text{ is closed iff } x = C(x)$$

...Top & CLOS

- Given a morphism in the closure system

$$f : (T, c) \to (T', c')$$

f is continuous for the associated topological spaces.

$$\Downarrow$$

Top is a *full subcategory* of **CLOS**.

Review: Examples

Example

Let in the following $\Sigma = (S, TF)$ denote signatures and Φ denote sets of FOL formulae:

1. **SPres** is the category of strict presentation with:
   - objects: $< \Sigma, \Phi >$,
   - morphism $\sigma :< \Sigma, \Phi > \rightarrow < \Sigma', \Phi' >$ is a sig. morph. $\sigma : \Sigma \rightarrow \Sigma'$ such that $\sigma(\Phi) \subseteq \Phi'$.
2. **Pres** is the category of presentation with:
   - objects: $< \Sigma, \Phi >$,
   - morphism $\sigma :< \Sigma, \Phi > \rightarrow < \Sigma', \Phi' >$ is a sig. morph. $\sigma : \Sigma \rightarrow \Sigma'$ such that $\sigma(c_\Sigma(\Phi)) \subseteq c_{\Sigma'}(\Phi')$.
3. **Theo** is the category of theories with:
   - objects: $< \Sigma, \Phi >$, where $\Phi = c_\Sigma(\Phi)$.
   - morphism $\sigma :< \Sigma, \Phi > \rightarrow < \Sigma', \Phi' >$ is a sig. morph. $\sigma : \Sigma \rightarrow \Sigma'$ such that $\sigma(\Phi) \subseteq \Phi'$.

(Co)Cartesian morphisms

Definition

Let $\varphi : D \to C$ be a functor and $f : c \to c'$ a C-morphism.

1. Let $d' : \mathbf{D}(c')$, a D-morphism $g : d \to d'$ is **cartesian** of f and d' iff:
   - $\varphi(g) = f$
   - $\forall g' : d'' \to d'$ and $f' : \varphi(d'') \to c$ such that $\varphi(g') = f'; f$, there is a *unique morphism* $h : d'' \to d$ such that $\varphi(h) = f'$ and $g' = g; h$
2. Let $d : \mathbf{D}(c)$, a D-morphism $g : d \to d'$ is **co-cartesian** of f and d iff:
   - $\varphi(g) = f$
   - $\forall g' : d \to d''$ and $f' : c' \to \varphi(d'')$ such that $\varphi(g') = f; f'$, there is a *unique morphism* $h : d' \to d''$ such that $\varphi(h) = f'$ and $g' = g; h$

(Co)Fibrations

Definition

Let $\varphi : D \to C$ be a functor

- φ is a **fibration** if, for every C-morphism $f : c \to c'$ and D-object d' in the fibre of c', there is a *cartesian morphism* for f and d'.
- φ is a **cofibration** if, for every C-morphism $f : c \to c'$ and D-object d in the fibre of c, there is a *co-cartesian morphism* for f and d'.

Specification as (Co)Fibrations

Example

Given a signature morphism $f : \Sigma \to \Sigma'$.

1. In **SPres**:
   - $f :< \Sigma, f^{-1}(\Phi') > \to < \Sigma', \Phi' >$ is a cartesian morphism for $< \Sigma', \Phi' >$.
   - $f :< \Sigma, \Phi > \to < \Sigma', f(\Phi') >$ is a co-cartesian morphism for $< \Sigma', \Phi' >$.
2. In **Pres**:
   - $f :< \Sigma, f^{-1}(c(\Phi')) > \to < \Sigma', \Phi' >$ is a cartesian morphism for $< \Sigma', \Phi' >$.
   - $f :< \Sigma, \Phi > \to < \Sigma', f(\Phi') >$ is a co-cartesian morphism for $< \Sigma', \Phi' >$.
3. In **Theo**:
   - $f :< \Sigma, f^{-1}(\Phi') > \to < \Sigma', \Phi' >$ is a cartesian morphism for $< \Sigma', \Phi' >$.
   - $f :< \Sigma, \Phi > \to < \Sigma', c(f(\Phi')) >$ is a co-cartesian morphism for $< \Sigma', \Phi' >$.

Cleavages, cloven fibrations

Definition

Let $\varphi : D \rightarrow C$ be a functor.
A *choice* of a *cartesian morphism* for every C-morphism $f : c \rightarrow \varphi(d')$ and for every D-object d' is called a **cleavage**.

A fibration equipped with a cleavage is called **cloven**.

Functor between fibres

Proposition

Let $\varphi : D \to C$ be a cloven fibration and $f : c \to c'$ a C-morphism.

1. *The morphism f defines a functor $\mathbf{f^{-1}} : D(c') \to D(c)$ as follows:*
   - *Given $d : D(c')$, $\mathbf{f^{-1}}(d')$ is the source of the Cartesian morphism $\varphi_{f,d'} : d \to d'$ that the cleavage associates with the fibration.*
   - *Given $g : d_1 \to d_2$ in $D(c'), \mathbf{f^{-1}}(g)$ is the morphism $\mathbf{f^{-1}}(d_1) \to \mathbf{f^{-1}}(d_2)$ that results from the universal property of the Cartesian morphism $\varphi_{f,d_2} : \mathbf{f^{-1}}(d_2) \to d_2$ when applied to $\varphi_{f,d_1}; g$ and id_c.*

Dualized notion

Definition

Let $\varphi : D \rightarrow C$ be a functor.
A *choice* of a *co-cartesian morphism* for every C-morphism $f : c \rightarrow \varphi(d')$ and for every D-object d is called a **cocleavage**.

A *cofibration* equipped with a cleavage is called **cloven**.

Proposition

Let $\varphi : D \rightarrow C$ be a cloven cofibration and $f : c \rightarrow c'$ a C-morphism.
The morphism f defines a functor $\mathbf{f} : D(c) \rightarrow D(c')$ in the dual way, i.e. by working on the target side of the co-cartesian morphism.

What if $f = id_c$ or $f = f_1; f_2$?

Proposition

Let $\varphi : D \to C$ be a functor.

1. *Given a C-object c and an object d in the fibre of c, the identity id_d is both Cartesian and co-Cartesian morphism for id_c and d.*
2. *Given C-morphisms $f_1 : c_1 \to c_2$ and $f_2 : c_2 \to c_3$,*
    - *an object d in the fibre of c_1, and*
    - *Cartesian morphisms $g_1 : d \to f_1(d)$ and $g_2 : f_1(d) \to f_2(f_1(d))$,*

    the composition $g_1; g_2$ provides a Cartesian morphism for $f_1; f_2$ and d.

Split Fibration

Definition

Let $\varphi : D \to C$ be a *cloven fibration*.
If for every C-object c,

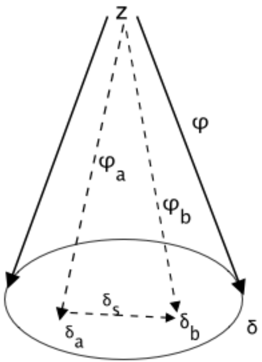
- id_c^{-1} is $id_{D(c)}$ and,
- for every decomposition $f = f_1; f_2$, f^{-1} is the composition $f_2^{-1}; f_1^{-1}$

then the fibration is said to be **split**.

Review1: Cone

Definition

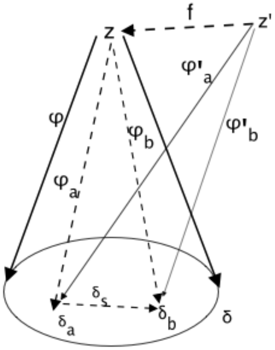
Let $\delta : I \rightarrow C$ be a diagram in category C. A *commutative cone* with base δ is an object z of C together with a family $\{\varphi_a : z \rightarrow \delta_a\}_{a \in I}$ of morphisms of C. And for every arrow $s : a \rightarrow b$ of graph I, $\varphi_a; \delta_s = \varphi_b$



Review2: Limit

Definition

Let $\delta : I \rightarrow C$ be a diagram in category C. A limit of δ is a *commutative cone* $\phi : z \rightarrow \delta$ such that, for every *commutative cone* $\phi' : z' \rightarrow \delta$, there is a unique morphism $f : z' \rightarrow z$ such that $\forall a \in I.\ f ; \varphi_a = \varphi'_a$.



Fibre Completness

Definition

A *cloven fibration* $\varphi : D \rightarrow C$ is said to be fibre-complete if:

- its fibres are complete categories
- the inverse translation functors induced on the fibres preserve limits.

Lifting limits

Proposition

Let $\varphi : D \rightarrow C$ be a split fibration.

1. *If φ is fibre-complete, then it lifts limits.*
2. *If in addition ϕ is amnestic, the lift is unique.*

Proof outline

Consider a diagram $\delta : I \to D$ in D.

- Study the diagram $\delta ; \varphi$ in C.
- As C is *complete*, exists a limit (c, μ) for $\delta ; \varphi$.
- Translate δ into the fibre $D(c)$.
- As fibres are required to be complete, there exists a limit (d, β) for $\delta ; \mu^{-1}$.
- $(d, \beta ; \varphi_{\mu_j, d_j})$ is a commutative cone for δ .

So far, we have constructed a commutative cone for δ .

Claim: This cone is a limit in D.

...proof outline

Consider a commutative cone (d', α) for δ .

- $(\varphi(d'), \varphi(\alpha))$ is a commutative cone for $\delta; \varphi$ in C.
- As (c, μ) is a limit for $\delta; \varphi$, exists a unique $f : \varphi(d') \rightarrow c$ such that $f; \mu = \varphi(\alpha)$.
- Translate :
  - the diagram $\delta; \mu^{-1}$
  - commutative cone (d, β)

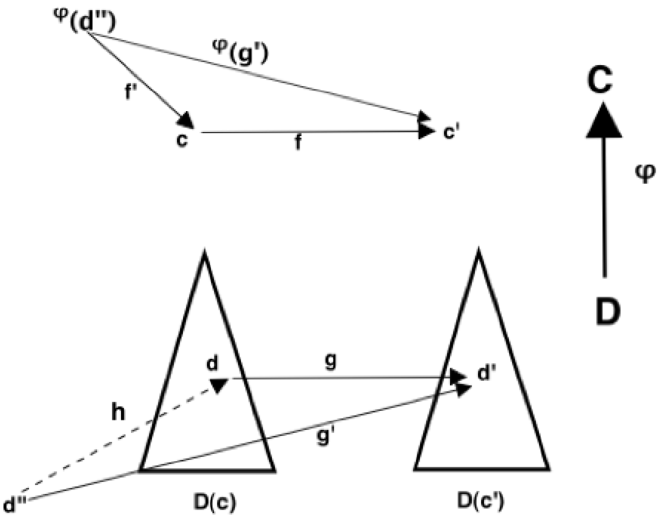
  into the fibre $D(\varphi(d'))$, using the functor f^{-1}.
- As inverse translation functors are required to preserve limits, $(f^{-1}(d), f^{-1}(\beta))$ is a limit for $\delta; \mu^{-1}; f^{-1}$.

So far, we have translated the diagram δ and the limit (d, β) into $\varphi(d')$ fibre.

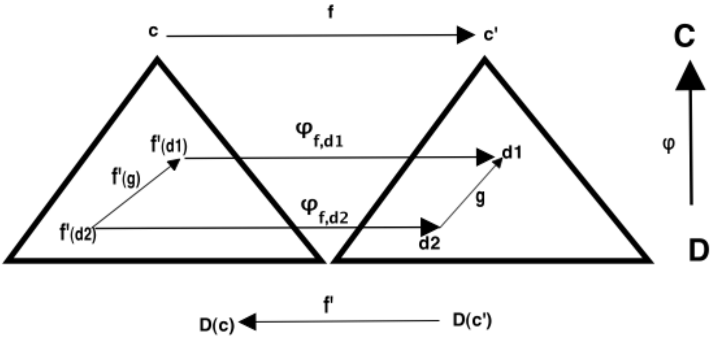
... proof outline

- Construct a commutative cone (d', α') for $\delta; \mu^{-1}; f^{-1}$.
  - commutative?
- As $(f^{-1}(d), f^{-1}(\beta))$ is a limit, there exists a unique morphism $h : d' \to f^{-1}(d)$ with $h; f^{-1}(\beta) = \alpha'$.
- Consider $h; \varphi_{f,d}$
  - $(h; \varphi_{f,d}); \beta; \varphi_{\mu_i, d_i} = \alpha_i$
  - $h; \varphi_{f,d}$ is unique.
    - Uniqueness?

Cartesian Morphism



Inverse functor



References

José Luiz Fiadeiro.
Categories for Software Engineering.
Springer-Verlag, Germany, 2005.

George E. Strecker Horst Herrlich.
Category Theory.
Allyn and Bacon Inc, Boston, 1973.

George E. Strecker Jiří Adámek, Horst Herrlich.
Abstract and concrete categories (the joy of cats).
Published under the GNU Free Documentation License, January 2004.

Saunders Mac Lane.
Categories for the Working Mathematician.
Springer-Verlag, New York, second edition, 1998.