Institutions - Part 2

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16.05.07

Outline

Outline

Every Institution Presents A Π-Institution

Theorem

Every institution $\langle \textbf{SIGN}, \textbf{gram}, \textbf{mod}, \models \rangle$ *presents the* π*-institution* $\langle \textbf{SIGN}, \textbf{gram}, \rightsquigarrow \rangle$*,*
where for every signature Σ, $p \in \textbf{gram}(\Sigma)$ *and* $\Phi \subseteq \textbf{gram}(\Sigma)$, $\Phi \rightsquigarrow_\Sigma p$ *iff for every* $M \in \textbf{mod}(\Sigma)$, $M \models_\Sigma \Phi$ *implies* $M \models_\Sigma p$*.*

Proof Outline

In order to prove this we must prove:

1. For every $p \in \textbf{gram}(\Sigma)$, $p \leadsto_\Sigma p$.
2. For every $p \in \textbf{gram}(\Sigma)$ and $\Phi_1, \Phi_2 \subseteq \textbf{gram}(\Sigma)$, if $\Phi_1 \subseteq \Phi_2$ and $\Phi_1 \leadsto_\Sigma p$ then $\Phi_2 \leadsto_\Sigma p$.
3. For every $p \in \textbf{gram}(\Sigma)$ and $\Phi_1, \Phi_2 \subseteq \textbf{gram}(\Sigma)$, if $\Phi_1 \leadsto_\Sigma p$ and for every $p' \in \Phi_1$, $\Phi_2 \leadsto_\Sigma p'$ then $\Phi_2 \leadsto_\Sigma p$.
4. For every $\sigma : \Sigma \to \Sigma'$, $p \in \textbf{gram}(\Sigma)$ and $\Phi \subseteq \textbf{gram}(\Sigma)$, $\Phi \leadsto_\Sigma p$ implies $\textbf{gram}(\sigma)(\Phi) \leadsto_{\Sigma'} \textbf{gram}(\sigma)(p)$.

Proof - Part 4

To show:

For every $\sigma : \Sigma \to \Sigma'$, $p \in \mathbf{gram}(\Sigma)$ and $\Phi \subseteq \mathbf{gram}(\Sigma)$, $\Phi \rightsquigarrow_\Sigma p$ implies $\mathbf{gram}(\sigma)(\Phi) \rightsquigarrow_{\Sigma'} \mathbf{gram}(\sigma)(p)$.

Definition of $\rightsquigarrow$

$\Phi \rightsquigarrow_\Sigma p$ iff for every $M \in \mathbf{mod}(\Sigma)$, $M \models_\Sigma \Phi$ implies $M \models_\Sigma p$.

Satisfaction Condition

For every $\sigma : \Sigma \to \Sigma'$, $p \in \mathbf{gram}(\Sigma)$ and $M' \in \mathbf{mod}(\Sigma')$, $\mathbf{mod}(\sigma)(M') \models_\Sigma p$ iff $M' \models_{\Sigma'} \mathbf{gram}(\sigma)(p)$.

Outline

Theories in Closure Systems

Given a closure system $\langle L, c \rangle$

- We say $\Phi \subseteq L$ is closed iff $\Phi = c(\Phi)$.
- We define the category $\mathbf{THEO}_{\langle L,c \rangle}$ whose objects (theories) are the closed subsets of L and morphisms are given by inclusions.
- We define the category $\mathbf{PRES}_{\langle L,c \rangle}$ whose objects (theories presentations) are the subsets of L and morphisms are given by the preorder $\Phi \leq \Gamma$ iff $c(\Phi) \subseteq c(\Gamma)$.
- We define the category $\mathbf{SPRES}_{\langle L,c \rangle}$ whose objects (strict presentations) are the subsets of L ordered by inclusion.

Outline

Every Π-Institution Presents An Institution

Theorem

Every π-institution $\langle \textbf{SIGN}, \textbf{clos} \rangle$ presents the institution $\langle \textbf{SIGN}, \textbf{gram} = \textbf{clos}; \textbf{forget}, \textbf{mod}, \models \rangle$, where for every signature Σ, $p \in \textbf{gram}(\Sigma)$ and $\Phi \in \textbf{mod}(\Sigma)$

$$\Phi \models p \text{ iff } p \in \Phi$$

We have to prove that the satisfaction condition holds.

Satisfaction Condition

For every $\sigma : \Sigma \rightarrow \Sigma'$, $p \in \textbf{gram}(\Sigma)$ and $M' \in \textbf{mod}(\Sigma')$, $\textbf{mod}(\sigma)(M') \models_\Sigma p$ iff $M' \models_{\Sigma'} \textbf{gram}(\sigma)(p)$.

Outline

Many Notions, Many Names and Confusion

Fiaderio	Π-Inst	Inst Map
Goguen		Inst Comorphism
Meseguer	Entailment System	Plain Map
Mossakowski		Plain Representation
Tarlecki		Representations

> *We had originally hoped to survey and systematise all the distinct notions of morphism that apply to the close variants of institutions; although we found even this limited goal impractical.*

Outline

Definition - Natural Transformations

Definition

Given two functors $\psi : \mathbf{D} \to \mathbf{C}$ and $\varphi : \mathbf{D} \to \mathbf{C}$, a natural transformation τ from ψ to φ, denoted by $\psi \xrightarrow{\tau} \varphi$ or $\tau : \psi \stackrel{\cdot}{\to} \varphi$, is a function that assigns to each object d of $\mathbf{D}$ a morphism $\tau_d : \psi(d) \to \varphi(d)$ of $\mathbf{C}$ such that, for every morphism $f : d \to d'$ of $\mathbf{D}$,

$$\tau_d ; \varphi(f) = \psi(f) ; \tau_{d'}$$

Outline

Definition - Institution Morphism

Definition

Let $\iota = \langle \mathbf{SIGN}, \mathbf{gram}, \mathbf{mod}, \models \rangle$ and $\iota' = \langle \mathbf{SIGN}', \mathbf{gram}', \mathbf{mod}', \models' \rangle$ be institutions. An institution morphism $\rho : \iota \to \iota'$ is a triple $\langle \boldsymbol{\Phi}, \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle$ where:

- $\boldsymbol{\Phi} : \mathbf{SIGN} \to \mathbf{SIGN}'$ is a functor.
- $\boldsymbol{\alpha} : \boldsymbol{\Phi}; \mathbf{gram}' \to \mathbf{gram}$ is a natural transformation.
- $\boldsymbol{\beta} : \mathbf{mod} \to \boldsymbol{\Phi}; \mathbf{mod}'$ is a natural transformation.

such that the following property(the invariance condition) holds for any signature $\Sigma \in |\ \mathbf{SIGN}\ |$, $m \in |\ \mathbf{mod}(\Sigma)\ |$ and $\phi' \in \mathbf{gram}'(\boldsymbol{\Phi}(\Sigma))$:

$$m \models_\Sigma \boldsymbol{\alpha}_\Sigma(\phi') \text{ iff } \boldsymbol{\beta}_\Sigma(m) \models'_{\boldsymbol{\Phi}(\Sigma)} \phi'$$

Definition - Institution Map (Comorphism)

Definition

Let $\iota = \langle \mathbf{SIGN}, \mathbf{gram}, \mathbf{mod}, \models \rangle$ and $\iota' = \langle \mathbf{SIGN}', \mathbf{gram}', \mathbf{mod}', \models' \rangle$ be institutions. An institution morphism $\rho : \iota \to \iota'$ is a triple $\langle \mathbf{\Phi}, \alpha, \beta \rangle$ where:

- $\mathbf{\Phi} : \mathbf{SIGN} \to \mathbf{SIGN}'$ is a functor.
- $\alpha : \mathbf{gram} \to \mathbf{\Phi}; \mathbf{gram}'$ is a natural transformation.
- $\beta : \mathbf{\Phi}; \mathbf{mod}' \to \mathbf{mod}$ is a natural transformation.

such that the following property(the invariance condition) holds for any signature $\Sigma \in\mid \mathbf{SIGN} \mid, m' \in\mid \mathbf{mod}'(\mathbf{\Phi}(\Sigma)) \mid$ and $\phi \in \mathbf{gram}(\Sigma)$:

$$\beta_\Sigma(m') \models_\Sigma \phi \text{ iff } m' \models'_{\mathbf{\Phi}(\Sigma)} \alpha_\Sigma(\phi)$$

Outline

CASL Specification subsorts.casl

```
spec subsorts =
  sort Nat

  ops 0,1 : Nat

  sort Pos = { x:Nat . not(x=0)}
  op pre : Pos -> Nat;
     suc : Nat -> Pos

  op one:Pos

axiom
forall n:Nat . pre(suc(n))=n;
1=one;
```

Mapping $PFOL^=$ to $FOL^=$

A $PFOL^=$-signature $\Sigma = (S, TF, PF, P)$ is translated to a $FOL^=$-presentation having

Signature

$$Sig(\Phi(\Sigma)) = (S, TF \uplus PF \uplus \{\bot : s \mid s \in S\}, P \uplus \{D : s \mid s \in S\})$$

Set of axioms $Ax(\Phi(\Sigma))$

$$\begin{array}{lll}
\exists x : s \bullet D_s(x) & s \in S & (1) \\
\neg D_s(\bot_s) & s \in S & (2) \\
D_s(f(x_1, \ldots, x_n)) \Leftrightarrow \bigwedge D_{s_i}(x_i) & f : s_1 \ldots s_n \rightarrow s \in TF & (3) \\
D_s(g(x_1, \ldots, x_n)) \Rightarrow \bigwedge D_{s_i}(x_i) & g : s_1 \ldots s_n \rightarrow s \in PF & (4) \\
p(x_1, \ldots, x_n) \Rightarrow \bigwedge D_{s_i}(x_i) & p : s_1 \ldots s_n \in P & (5)
\end{array}$$

CASL Specification partial.casl

```
spec partial =
  sort Nat

  ops 0,1 : Nat

  op pre : Nat ->? Nat;
     suc : Nat -> Nat

axiom
forall n:Nat . pre(suc(n))=n;
```

Outline

Proposition for Institution Morphism

Let $\rho = \langle \mathbf{\Phi}, \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle : \iota \rightarrow \iota'$ be an institution map. The functor $\mathbf{\Phi}$ extends to $\mathbf{THEO}_{\iota} \rightarrow \mathbf{THEO}_{\iota'}$ by establishing $\mathbf{\Phi}(\langle \Sigma, \Gamma \rangle) = \langle \mathbf{\Phi}(\Sigma), c(\boldsymbol{\alpha}_{\Sigma}(\Gamma)) \rangle$.

Proposition for Institution Map

Let $\rho = \langle \boldsymbol{\Psi}, \alpha', \beta' \rangle : \iota \to \iota'$ be an institution morphism. The functor $\boldsymbol{\Psi}$ extends to $\mathbf{THEO}_{\iota'} \to \mathbf{THEO}_{\iota}$ through $\boldsymbol{\Psi}(\langle \Sigma', \Gamma' \rangle) = \langle \boldsymbol{\Psi}(\Sigma'), \boldsymbol{\alpha'}^{-1}_{\Sigma'}(\Gamma') \rangle$.

Proposition

Let $\iota = \langle \mathbf{SIGN}, \mathbf{gram}, \mathbf{mod}, \models \rangle$ and $\iota' = \langle \mathbf{SIGN}', \mathbf{gram}', \mathbf{mod}', \models' \rangle$ be institutions, $\rho = \langle \boldsymbol{\Phi}, \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle : \iota \to \iota'$ be an institution map and $\langle \boldsymbol{\Psi}, \boldsymbol{\alpha}', \boldsymbol{\beta}' \rangle : \iota \to \iota'$ be a morphism such that ψ is a right adjoint of ϕ, and, for every $\Sigma \in | \mathbf{SIGN} |$, $\boldsymbol{\alpha}_\Sigma = \mathbf{gram}(\eta_\Sigma); \alpha'_{\Phi(\Sigma)}$ where η is the unit of the adjunction, then

- The functor $\boldsymbol{\upsilon} : \mathbf{THEO}_{\iota'} \to \mathbf{THEO}_\iota$ induced by the institution morphism $\langle \boldsymbol{\Psi}, \boldsymbol{\alpha}', \boldsymbol{\beta}' \rangle$, is a right adjoint of the functor $\mathbf{THEO}_\iota \to \mathbf{THEO}_{\iota'}$ induced by the institution map $\langle \boldsymbol{\Phi}, \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle$.
- If each component of $\boldsymbol{\beta}'$ is surjective, then the units η_Σ are conservative.

Summary

- Institutions provide a frame work for dealing with logics, that capture the notions of sentences, model and satisfaction between models and sentences.
- Institution morphisms and Comorphisms allow us to translate between institutions, which allow us to use programs on one logic with another logic.
- They actually have a practical use in the real world. eg. Hets.