Various Logics Fit into the Framework of an Algebraic Logic

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Contents

Recap

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What is a Signature?

$$\Sigma = (S, \Omega)$$

Operations are total functions of the form

$$n : s_1 \times \ldots \times s_k \rightarrow s$$

with $s_1, \ldots, s_k, s \in S$ and $k \geqslant 0$.

What is a Term?

Given a signature $\Sigma = (S, \Omega)$ and a set of variables X, then $T_{\Sigma(X),s}$ is defined by

1. $X_s \subseteq T_{\Sigma(X),s}$,
2. if $n :\rightarrow s$ is an operation of Ω then $n \in T_{\Sigma(X),s}$,
3. if $n : s_1 \times \ldots \times s_k \rightarrow s, k \geqslant 1$ is an operation of Ω and if $t_i \in T_{\Sigma(X),s_i}$, for $1 \leqslant i \leqslant k$, then $n(t_1, \ldots, t_k) \in T_{\Sigma(X),s}$.

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What is an Assignment?

Given $\Sigma = (S, \Omega)$ then an assignment of X for A is a family $\alpha = (\alpha_s)_{s \in S}$ of functions $\alpha_s : X_s \to A(s)$.

We can just write $\alpha : X \to A$.

How do we Evaluate Terms?

Given a signature $\Sigma = (S, \Omega)$, a Σ-Algebra A, a set of variables X, a term $t \in T_{\Sigma(X)}$ and an assignment $\alpha : X \to A$ then $A(\alpha)(t)$ is defined by:

1. $A(\alpha)(t) = \alpha_s(x)$ if $t = x$ with $x \in X_s, s \in S$,
2. $A(\alpha)(t) = A(w)$ if $t = n$ and $w = (n :\to s) \in \Omega$,
3. $A(\alpha)(t) = A(w)(A(\alpha)(t_1), \ldots, A(\alpha)(t_k))$
   if $t = n(t_1, \ldots, t_k)$,
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Motivation

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There are many logics.

Equational Logic Example - Monoid

```
spec Monoid =
  sort M;

  ops e: M;
      __+__ : M * M -> M;

  forall x, y, z : M
  . (x + y) + z = x + (y + z)
  . x + e = x
  . e + x = x
end
```

First Order Logic Example - Boolean

Note: $(\varphi_1 \vee \varphi_2)$ is an abbreviation for $(\neg((\neg\varphi_1) \wedge (\neg\varphi_2)))$.

```
spec Boolean =
  sort Bool;

  ops TRUE, FALSE : Bool;
      __ AND __ : Bool * Bool -> Bool;

  forall x, y : Bool
  . x = TRUE \/ x = FALSE
  . not TRUE = FALSE
  . x AND y = FALSE when x = FALSE else y
end
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Logics

The Set of Formulae for EL

For each signature Σ the set of formulae of EL is

$$EL(\Sigma) = \{\forall X.t = u \mid t, u \in T_{\Sigma(X),s}\}.$$

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forall x, y, z : M
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The Satisfaction Relation for EL

Let Σ be a signature.

$$A \models_\Sigma \forall X.t = u$$

:iff

for all assignments $\alpha : X \to A, A(\alpha)(t) = A(\alpha)(u)$

for each Σ-Algebra A and
for each equation $\forall X.t = u \in EL(\Sigma)$.

Why Different Logics - Theorem

Theorem: Specifications in EL always have the one point model.

Why Different Logics - Proof

Let Σ be a signature, A be a Σ-Algebra with $\mid A_s \mid = 1$

Let $A_s = \{*_s\}$

$A(\alpha)(t) \in A_s$ for all assignments $\alpha : X \to A$

$A(\alpha)(t) = *_s$

$A \models_\Sigma \varphi$

$\Leftrightarrow A \models_\Sigma (\forall X.t = u)$

$\Leftrightarrow A(\alpha)(t) = A(\alpha)(u)$ for all assignments $\alpha : X \to A$

$\Leftrightarrow *_s = *_s$

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Why Different Logics - Proof

Let Σ be a signature, A be a Σ-Algebra with $| A_s |= 1$
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Why Different Logics - Corollary

- The Booleans cannot be specified in EL.
- Reminder: The Booleans can be specified in FOL (See example earlier).

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Algebraic Logics

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What's needed for an Algebraic Logic?

- A Logic,
- The Isomorphism condition must hold,
- . . .

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Isomorphism Condition

Let Σ be a signature,
Let φ be a formula,
Let A, B be Σ-algebras with $A \simeq B$

$$A \models_{\Sigma} \varphi \text{ iff } B \models_{\Sigma} \varphi$$

EL Satisfies the Isomorphism Condition

Let Σ be a signature, A, B be Σ-Algebras, $A \simeq B$, and $\forall X.t = u$ be a formula in $EL(\Sigma)$. we have to show:

$$A \models_\Sigma \forall X.t = u \Leftrightarrow B \models_\Sigma \forall X.t = u$$

Then we know that $A(\alpha)(t) = A(\alpha)(u)$ for all assignments $\alpha : X \rightarrow A$.

We have to show that $B(\beta)(t) = B(\beta)(u)$ for all assignments $\beta : X \rightarrow B$.

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EL Satisfies the Isomorphism Condition (2)

Let $\beta : X \to B$ be an assignment, $h : A \to B$ be the isomorphism.

$$\begin{aligned}
B(\beta)(t) &= h(h^{-1}(B(\beta)(t))) && \text{by isomorphism identity} \\
&= h(A(h^{-1} \circ \beta)(t)) && \\
&= h(A(h^{-1} \circ \beta)(u)) && \text{since } h^{-1} \circ \beta : X \to A \\
&= h(h^{-1}(B(\beta)(u))) && \\
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Hence equational logic satisfies the isomorphism condition.

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- There are many logics that fit in to the framework of an Algebraic Logic,
- A logic has to satisfy the Isomorphism Condition to be an Algebraic Logic,
- EL, CEL, FOL are Algebraic Logics,
- FOL can express more than EL.

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References

Material has been laid out in the same way as done in the book **Specification of Abstract Data Types** *J. Loeckx, H.Ehrich and M.Wolf 1996*