3D Symmetric, Traceless Tensor Field Analysis and Visualization

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Fig. 1. In the Clouds: Vancouver from Cypress Mountain

Abstract—Duis autem vel eum iriure dolor in hendrerit in vulputate velit esse molestie consequat, vel illum dolore eu feugiat nulla facilisi at vero eros et accumsan et justo odio dignissim qui blandit praesent luptatum zzril delenit augue duis dolore te feugait nulla facilisi. Lorem ipsum dolor sit amet, consectetur adipiscing elit, sed diam nonummy nibh euismod tincidunt ut laoreet dolore magna aliquam erat volutpat.

And this is what references look like [?].

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Index Terms—Radiosity, global illumination, constant time.

1 INTRODUCTION

Three-dimensional symmetric tensor fields have a wide range of applications in physics, chemistry, engineering, and medicine. For example, stress and strain tensors can be used to study solid and fluid mechanics as well as material properties. There has been much work in visualizing 3D symmetric tensor fields, primarily for the medical applications such as brain imaging. The focus of the visualization under such settings is the ability to track hyperstreamlines, i.e., lines that following one of the eigenvector directions (typically the major eigenvector field). Various forms of anisotropy have also been defined and visualized along with the hyperstreamlines. In some more recent work, Zheng and et al. [? ] study the degenerate features in 3D symmetric tensor fields from a topological point of view. This led to a number of improved method to detect degenerate points (which form curves). In addition, they study tensor behaviors near degenerate curves and point out that they exhibit 2D degenerate tensor patterns such as wedges and trisectors. Their work is primarily based on the analysis of a discriminant function.

Zheng et al.’s pioneering work is particularly inspirational to our research. In this paper, we seek to expand the horizon on 3D symmetric tensor field analysis and visualization in a number of directions.

1. We parameterize the space of 3D symmetric tensor fields which can provide a more global view of the set of such tensor fields. Our analysis is based on the well-known isotropic-deviator decomposition. However, by studying traceless tensors, we show that much the analysis can be simplified and become more geometrically intuitive.

2. We reformulate the problem of finding degenerate curves as finding the zero levelset of two cubic polynomials. In contrast to the discriminant function of Zheng and Pang, which is of degree six, our descriptor makes the extraction of degenerate curves more robust. Moreover, it enables us to compute the curvature and torsion of degenerate curves, which to the best of our knowledge, were not addressed.

3. We point out the boundary surface between linear degeneracy and planar degeneracy is important to the understanding of tensor fields and include them as part of tensor field topology. Furthermore, we point out the physical interpretations of the normal and curvature tensors on this surface.

4. We provide some additional results for piecewise linear tensor fields defined on tetrahedral meshes.

5. We provide visualization based on our analysis.
6. We apply our analysis to a number of applications and provide physical interpretations.

2 Previous Work

3 Tensor Background

We review relevant facts about $3 \times 3$ tensors as well as 3D tensor fields.

A $3 \times 3$ tensor is a matrix $T_{ij}$ ($1 \leq i, j \leq 3$). $T$ is symmetric if $T_{ij} = T_{ji}$ for any $1 \leq i, j \leq 3$. This paper focuses on symmetric tensors. Consequently, in the remainder of the paper we will drop the mention of symmetric when referring to symmetric tensors. $\lambda \in \mathbb{R}$ is an eigenvalue if there exists a non-zero vector $v$ such that $Tv = \lambda v$. In this case $v$ is an eigenvector. In this, $k$ for any non-zero $k \in \mathbb{R}$ is also an eigenvector.

T has three eigenvalues $\lambda_1 \geq \lambda_2 \geq \lambda_3$. $T$ is degenerate if at least two of the eigenvalues are the same. A degenerate tensor can be either a linear degeneracy ($\lambda_1 > \lambda_2 = \lambda_3$), or a planar degeneracy ($\lambda_1 > \lambda_2 > \lambda_3$), or a triple degeneracy ($\lambda_1 = \lambda_2 = \lambda_3$). The union of linear degeneracies and planar degeneracies are referred to as double degeneracies. For double degeneracies the eigenvectors belonging to the repeating eigenvalue form a plane, which is referred to as the degenerate plane. The normal to this plane is an eigenvector of the third (and different) eigenvalue, which we refer to as the non-degenerate eigenvalue.

There are four important quantities derived from $T$ that are invariant under the change of basis: (1) trace: $\mathbf{P} = \lambda_1 + \lambda_2 + \lambda_3$, (2) minor $\mathbf{Q} = \lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1$, (3) determinant $\mathbf{R} = \lambda_1 \lambda_2 \lambda_3$, and (4) tensor magnitude: $||T|| = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$. A degenerate tensor can be described as the zeros of the so-called discriminant

$$
D = (\lambda_1 - \lambda_2)^2(\lambda_2 - \lambda_3)^2(\lambda_3 - \lambda_1)^2\quad (1)
$$

$$
= Q^3P^2 - 4RP^3 - 4Q^3 + 18PQR - 27R^2\quad (2)
$$

A tensor field is continuous, tensor-valued function in $\mathbb{R}^3$. A point $p$ is a (linear, planar, triple) degenerate point if $T(p)$ is a degenerate tensor of a corresponding type. The set of triple degenerate points is numerically unstable, while the set of both linear and planar degenerate points form curves in a numerically stable fashion. The tangent direction of a degenerate curve at a point $p$ is the direction in which the projected tensor field onto the degenerate plane at $p$ remains degenerate. Furthermore, the projected 2D tensor exhibit 2D degenerate patterns such as wedges and trisectors. Along a degenerate curve, the type of 2D degenerate pattern persists except at the switching points, where the tangent to the degenerate curve is inside the degenerate plane. In these cases, the 2D tensor patterns can switch from a wedge to a trisector, and vice versa.

4 Mathematical Analysis

At the core of our analysis is the well-known isotropic-deviator decomposition, which allows us to greatly simplify the representation and analysis for tensor fields. Moreover, it brings additional insight, especially geometric intuition and physical interpretation that were not available when inspecting the whole tensors, i.e., not necessarily traceless.

$$
T = D + S \quad (3)
$$

where $D = \frac{P}{3}I$ and $S = T - D$ are the isotropic part and deviator of $T$, respectively. Notice that the eigenvalues of $S$, $\lambda_i's$, can be obtained from the eigenvalues of $T$, $\lambda_i's$, as follows:

$$
\lambda_i's = \lambda_i - \frac{P}{3} \quad (4)
$$

Consequently, $\text{trace}(S) = \text{trace}(T) - \frac{2P}{3}$, i.e., $S$ is traceless. In fact, eigenvector information of $T$ is contained in $S$ if the sense that a vector $v$ is an eigenvector of $T$ if and only if it is also an eigenvector in $S$. In contrast, any non-zero vector is an eigenvector of $D$, i.e., directionless or isotropic. In addition, $T$ is triply, linearly, or planarly degenerate if and only $S$ is of the same type of degeneracy. These facts suggest that to perform analysis of a tensor field in terms of its anisotropy, it is sufficient to study the deviator tensor field.

The set of 3D traceless tensors is a five-dimensional linear space, that is, the sum of two traceless tensors as well as a scalar multiple of a traceless tensor are both traceless. Such tensors have the following form:

$$
\begin{pmatrix}
    a & b & c \\
    b & d & e \\
    c & e & f
\end{pmatrix}
$$

where $f = -a - d$. The discriminant of traceless tensors, i.e., $P = 0$, have a much reduced form:

$$
D = -4Q^3 - 27R^2\quad (5)
$$

Moreover, it is straightforward to verify that $Q = -\frac{||P||}{2}$. The above formula now becomes

$$
D = 4||T||^3 - 27R^2\quad (6)
$$

We now consider all non-zero traceless tensors. There are two types of degeneracies: linear and planar. We say that a non-degenerate tensor to be an L-type if $\lambda_1 - \lambda_2 > \lambda_2 - \lambda_3$. Similarly, a non-degenerate tensor is a P-type if $\lambda_1 - \lambda_2 < \lambda_2 - \lambda_3$. This definition leads to a partition of the domain into the $L$-region and $P$-region. The boundary surface between the $L$- and $P$-regions are referred to as purely non-generate tensors, due to the fact that such tensors would take the most modifications in order to become degenerate. This inspires the following integral formula for degeneracy:

$$
G = \frac{(2\lambda_1 - \lambda_2 - \lambda_3)(2\lambda_2 - \lambda_1 - \lambda_3)(2\lambda_3 - \lambda_1 - \lambda_2)}{||T||^{3/2}}\quad (7)
$$

A tensor is purely non-degenerate if and only if the above formula evaluates to zero. Interestingly, for traceless tensors, the above formula reduces to $\frac{27R^2}{||T||^3}$. Notice that $\lambda_1 > 0$ and $\lambda_3 < 0$ for traceless tensors, which means that the sign of $\text{sign}(R) = \text{sign}(\lambda_3)$. In other words, $T$ is in the $P$-region, the $L$-region, or on the boundary surface between the $L$- and $P$-regions, if $\lambda_3$ is positive, negative, or zero, respectively.

It can be verified that the maximum and minimum of $G = \frac{27R^2}{||T||^3}$ are reached when $T$ is a planar degenerate tensor and a linear degenerate tensor, respectively. We can show that the optimum of $G$ on a level set $\sqrt{||T||} = m > 0$ satisfy that $\nabla R \times \nabla ||T|| = 0$. This gives two conditions that degenerate curves must satisfy:

$$
\frac{dR}{dy} \frac{d||T||}{dx} - \frac{dR}{dx} \frac{d||T||}{dy} = 0 \quad (9)
$$

$$
\frac{dR}{dy} \frac{d||T||}{dz} - \frac{dR}{dz} \frac{d||T||}{dy} = 0 \quad (10)
$$

It can be verified that $\nabla R \times \nabla ||T|| = (\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)(\lambda_3 - \lambda_1) \nabla \lambda_1 \times \nabla \lambda_2 \times \nabla \lambda_3$. In a way, $\nabla R \times \nabla ||T||$ is the square root of the discriminant $D$, but is a vector instead of just a scalar. There are a number of advantages to use $\nabla R \times \nabla ||T|| = 0$ when finding degeneracies in a tensor field. First, instead of searching the root of a non-negative function, we can now search for the intersection of two surfaces. This allows us to provide analytical formulas for the curvature and torsion of degenerate curves, which was not available when given $D$. For piecewise linear tensor fields defined a tetrahedron mesh, $D$ is a degree six polynomial. In contrast, the conditions in the above equations are two cubic polynomials, which implies better numerical stabilities.

This leads to interesting observations. A degenerate curve basically connects points on the levelset surfaces of $||T||$ where the gradient of $R$ is parallel to the normal of the levelset. For a piecewise linear tensor
field, i.e., where the coefficients of $T(x,y,z)$ are linear functions of $x$, $y$, and $z$, it is straightforward to verify that the level sets of $||T||$ are quadratic surfaces with a positive semi-definite, i.e., ellipsoids, cylinders, or planes. Notice that the latter two cases are degenerate. In non-degenerate cases, the ellipsoids will be nested with the core point, i.e., the zeroth-levelset.

The above formula also allows the curvature and torsion for the degenerate curve to be computed.

5 ALGORITHMS

6 RESULTS

6.1 Earthquake Modeling

6.2 Flows

7 CONCLUSIONS

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