Set theoretical proofs as type theoretical programs

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We show, that all Π^0_2-sentences, provable in the set theory KPI^+_U can be proved in Martin-Löf's Type Theory with W-type and one universe. Therefore set theoretical proofs can be considered as programs in type theory. The method used is a formalisation of proof theoretical methods in type theory. The result will be a high level type theory program using the full strength of Martin-Löf's Type Theory. We suggest the use of Kripke-Platek style set theory as a programming language.

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1 Introduction

Mathematics is usually developed on the basis of set theory. When trying to use type theory as a new basis for mathematics, most of mathematics has to be reformulated. This is of great use, because then the step to programs is direct and one can expect to get the best programs. However, it seems that most mathematicians will continue to work in set theory. Even when changing to type theory for the formalisation, usually the proofs will be developed first having classical set theory in the background. Therefore methods for transferring directly set theoretical arguments to type theory could make the step from traditional mathematics to type theory and therefore to computer science far easier.

The reason why set theory is used in mathematics is its high flexibility and that it allows to write down expressions without having to care about the type of the object. Therefore, if set theoretical proofs can be transferred to type theory, one could use set theory as a programming language added to type theory.

In our definition of KPI_U^+ only natural numbers are included as urelemente, which form the basic data structure, for which programs can be extracted. However, the method used is not at all restricted to this particular structure. Lists and free algebras can be included easily and we are working on an extension to data structures of higher type.

Π_2^0-sentences can be considered as specifications of programs, and proofs in Martin-Löf's Type Theory are programs. In this abstract we will prove, that all Π_2^0-sentences provable in a certain set theory $KPI_U^+ = \bigcup_{n\in\mathbb{N}} KPI_{U\,n}^+$ can be proved in ML_1W, Martin-Löf's Type Theory with W-type and one universe. KPI_U^+ is **K**ripke **P**latek set theory with **u**relemente (the natural numbers), one admissible (admissible are the recursive analogue of cardinals) closed under the step to the next admissible (a recursive **i**naccessible) and finitely many admissibles above it. This works for all variations (intensional, extensional, different versions of the identity type). Since, in [Se93] we have shown, that all arithmetical sentences provable in ML_1W are theorems of KPI_U^+, it follows, that these two theories have the same Π_2-theorems. Therefore, transferring programs to ML_1W from proof theoretically stronger set theories is no longer possible.

The method used here is certainly feasible, the only exception is the well ordering proof, which will be used here, and seems to be too long for practical applications. However, one can think about conservative extensions of ML_1W by adding types, the elements of which represent ordinal denotations, and rules for transfinite induction. Then everything shown here can be easily implemented in Martin-Löf's Type Theory.

We use here techniques from proof theory. These are based on ordinal analysis. However, very basic knowledge about ordinals is sufficient for understanding this proof, since we are just formalising a proof, which need not be understood itself.

Our method is heavily built on transfinite induction. In [Se95] the author has shown, that ML_1W shows transfinite induction up to the ordinals $\psi_{\Omega_1}(\Omega_{I+n})$, therefore as well up to

$\alpha_n := \psi_{\Omega_1}(\epsilon_{\Omega_{I+n}+1}) + 1$. Transfinite induction up to α_n is exactly what we need in order to analyse $KPI^+_{U\,n}$. Now it is just necessary to formalise this analysis in ML_1W using, that we have transfinite induction up to α_n, and to extract the validity of Π^0_2-sentences from the cut free proofs.

This formalisation is not trivial, since in modern methods (like Buchholz' $\mathcal{H}$-controlled derivations), proof theoretical analysis is carried out in full set theory. However, using proof trees with a correctness predicate, we are able to overcome this difficulty.

The methods used here can on one hand extended to all recent proof theoretical studies using infinitary derivations and ordinal analysis. Only, the type theory is not available yet, except for Mahlo universes. (For Mahlo, the author presented a type theory on the Logic Colloquium '95 in Haifa, there is related work by Rathjen and Griffor). Further, one sees easily, that the well-foundedness of the W-type is not needed really here, since we have always a descent in ordinals. (For the RS^*-derivations, $||\Gamma||$ is descending). Therefore, by replacing the W-type by a recursive object obtained using the recursion theorem, (so I becomes now a not necessarily least fixed point – one naturally has to replace in the proof of lemma 7 f by a list coded as a natural number) which can be defined in PRA, one shows with nearly the same proof, that $PRA + TI(OT_n)$ shows all Π_2-sentences of KPI^+_U.

Independently, W. Buchholz has taken a different approach for obtaining the same result, by using denotation systems (extending [Bu91]). This has the advantage of giving directly executable programs, whereas our method has the advantage of being very perspicuous and explicit.

The other approach for extracting programs from classical proofs are based on the A-translation. This can even be carried out for full set theory, as shown by Friedman (a good presentation can be found in [Be85] section VIII.3). A lot of research is carried out for extracting practical programs using the A-translation, see for instance [BS95] or [Sch92]. However, since Martin-Löf's Type Theory is already a programming language, we believe, that our approach allows to switch more easier between classical proofs and direct programming. Further, in KPI^+_U one has constructions corresponding precisely to the different type constructors in type theory, so with our method we have good control over the strength of the methods used.

2 General Assumptions

Assumption 1 *(a) We assume some coding of sequences of natural numbers.* $< k_0, \ldots, k_l >$ *denotes the sequence* $k_0, \ldots, k_l$ *and* $(k)_i$ *the* i*-th element (beginning with* $i = 0$*) of the sequence* k*.*

(b) In the following we will omit the use of Gödel-brackets.

(c) Let $n_0 : N$ *be fixed.*

Definition 2 *(a) Let* OT *be defined as in [Se95], definition 3.9. We define* OT_{n_0} *by:*
$O, I \in OT_{n_0}$.
If $\alpha, \beta \in OT_{n_0}$, $\gamma =_{NF}' \alpha + \beta \vee \gamma =_{NF} \phi_\alpha\beta \vee \gamma =_{NF} \Omega_\gamma \vee \gamma =_{NF} \psi_\beta\gamma$, $\gamma \in$

$OT \cap \epsilon_{\Omega_{I+n_0}+1}$, then $\gamma \in OT_{n_0}$.
In the following α, β, γ denote elements of OT_{n_0}.

(b) *We restrict the ordering $\prec$ on OT to OT_{n_0} (replace $\prec$ by $\prec \cap OT_{n_0} \times OT_{n_0}$)*

(c) *Let ML_1W be Martin-Löf's Type Theory with W-type and one universe, as for instance formulated in [Se95], or any other formulation (for instance we can use the the identity type together with the elimination operator).*

(d) *For arithmetical sentences ϕ, let $\widehat{\phi}$ the canonical interpretation of ϕ in ML_1W.*

Theorem 3 *If $ML_1W \vdash n : N \Rightarrow \phi(n)$ type, then $ML_1W \vdash \forall k \in OT_{n_0}.((\forall l \prec k.\phi(k)) \rightarrow \phi(l)) \rightarrow \forall k \in OT_{n_0}.\phi(k)$.*

Proof: Let $\mathcal{W}' := \mathcal{W}_{n_0+1}$ as in [Se95], definition 5.37. Then $OT_{n_0} \subset \mathcal{W}'$. Let $\psi(x) := x \notin OT_{n_0} \vee \phi(x)$. Then $Prog(\mathcal{W}_0, (x)\psi(x))$, $\forall k \in \mathcal{W}_0.\psi(k)$, and the assertion.

3 The set theory $KPI^+_{U\,n}$

Definition 4 *of the theory Kpi^+_u*

(a) *The language of $KPI^+_{U\,n_0}$ consists of infinitely many number variables, infinitely many set variables, symbols for finitely many primitive recursive relations (on natural numbers) P of arbitrary arity, the relations Ad, $\overline{Ad}$, $\in$ and $\notin$ (the latter are written infix) and the logical symbols $\wedge, \vee, \forall, \exists$.*

In the following n, m denote number variables and a, b, c denote set variables, to which we might add (this will apply to all future such conventions) indices, tilde-symbol or accents.

an, bn, cn, am, bm, cm denote variables, which are either set variables or number variables.

We assume that that $P_=$, the 2-ary equalitybetween two natural numbers, $\perp$, the 0-ary false relation, and for every primitive recursive relation P, the negation of this relation $\overline{P}$ are in the set of primitive recursive relations. $\top := \overline{\perp}$.

(b) *Number terms are $S^k(0)$ and $S^k(n)$, where $k \in \mathbb{N}$, $S^0(r) := r$, $S^{k+1}(r) := S(S^k(r))$. sn and tn denote number terms. The set terms are the set variables.*

We define $val(S^k(0)) := k$.

(c) *Prime formulas are $P(tn_1, \dots, tn_k)$, where P is an k-ary primitive recursive relation symbol, $Ad(a)$, $\overline{Ad}(a)$, $s \in a$, $s \notin a$.*

(d) *Formulas are prime formulas, and if ϕ and ψ are formulas then $\phi \wedge \psi$, $\phi \vee \psi$, $\forall a.\phi$, $\forall n.\phi$, $\exists a.\phi$, $\exists n.\phi$ are formulas. ϕ, ψ denote formulas in the following.*

(e) *We define the negation of a formula by the deMorgan rules: $\neg P(tn_1, \dots, tn_k) := \overline{P}(tn_1, \dots, tn_k)$, $\neg(s \in a) := s \notin a$, $\neg Ad(a) := \overline{Ad}(a)$, $\neg(\psi \wedge \phi) := \neg(\psi) \vee \neg(\phi)$, $\neg(\forall an.\phi) := \exists an.\neg\phi$, $\neg(\neg(\phi)) := \phi$ otherwise.*

(f) $\forall an \in b.\phi := \forall an.an \in b \to \phi$, $\exists an \in b.\phi := \neg(\forall an \in b.\neg\phi)$.

(g) *A formula is arithmetical, if it contains neither set terms nor set variables,* Δ_0*, if it contains only restricted set-quantifiers.*
It is in Σ_1*, if it contains no unrestricted universal set quantifier, in* Σ_1^{arith}*, if it is arithmetical and contains no unrestricted universal number quantifier.*
The arithmetical Π_2^0*-formulas are formulas* $\forall n.\phi$, $\phi \in \Sigma_1^{arith}$.
$\phi \in \Sigma(\kappa) :\Leftrightarrow \phi = \psi^{L_\kappa}$ *for some* $\psi \in \Sigma_1$*. In this situation,* $\phi^{t,\kappa} := \psi^t$*. Further* $\phi^{\beta,\kappa} := \phi^{L_\beta,\kappa}$.
If ϕ *is a formula, let* ϕ^a *be the result of replacing in* ϕ *every unrestricted set-quantifier (not number quantifier)* $\forall b$ *by* $\forall b \in a$*, and* $\phi^\beta := \phi^{L_\beta}$.

(h) $\forall x.\phi := (\forall n.\phi[x := n]) \wedge (\forall a \in b.\phi[x := a])$*, where* $\phi[x := n]$ *is the result of substituting for* x n *in* ϕ*, if the result is a formula, and* $\phi[x := n] := \perp$ *otherwise, similar for* $\phi[a := n]$.

(i) Γ, Δ *denote multi-sets of formulas.* $\Gamma, \Delta := \Gamma \cup \Delta$, $\Gamma, \phi := \Gamma \cup \{\phi\}$.

(j) $\phi \to \psi := \neg\phi \vee \psi$, $\phi \iff \psi := (\phi \to \psi) \wedge (\psi \to \phi)$.

For number or set terms s, t *we define:*
$a \subset b := \forall x \in a.x \in b$.

$$s = t := \begin{cases} P_=(s,t) & \text{if } s,t \text{ are number terms} \\ s \subset t \wedge t \subset s \wedge & \\ \quad (Ad(s) \iff Ad(t)) & \text{if } s,t \text{ are set terms} \\ \perp & \text{otherwise} \end{cases}$$

$trans(a) := \forall b \in a.\forall x \in b.x \in a$.

(k) *The logical rules are* $\Gamma, \phi, \neg\phi$, $\frac{\Gamma,\phi \quad \Gamma\psi}{\Gamma\phi\wedge\psi}$, $\frac{\Gamma,\phi}{\Gamma\phi\vee\psi}$, $\frac{\Gamma,\psi}{\Gamma\phi\vee\psi}$, $\frac{\Gamma,\phi}{\Gamma,\forall n.\phi}$ *(if* $n \notin FV(\Gamma)$*)*, $\frac{\Gamma,\phi}{\Gamma,\forall a.\phi(a)}$ *(if* $a \notin FV(\Gamma)$*)*, $\frac{\Gamma,\phi[n:=tn]}{\Gamma,\exists n.\phi}$, $\frac{\Gamma,\phi[n:=a]}{\Gamma,\exists a.\phi}$*, and* $\frac{\Gamma,\phi \quad \Gamma\neg\phi}{\Gamma}$.

(l) *Axioms of* $KPI^+_{U\,n_0}$ *The set axioms are:*

(Ext_1)	$\forall x.\forall y.\forall a.(x = y \to x \in a \to y \in a)$.
(Ext_2)	$\forall a.\forall b.(a = b \to Ad(a) \to Ad(b))$.
$(Found)$	$\forall \vec{an}.[\forall a.(\forall b \in a.\phi(b, \vec{an}))\phi(a, \vec{an})] \to \forall a.\phi(a, \vec{an})$.
$(Pair)$	$\forall x, y.\exists a.x \in a \wedge y \in a$.
$(Union)$	$\forall a.\exists b.\forall y \in a.\forall x \in y.(x \in b)$.
$(\Delta_0 - Sep)$	$\forall \vec{an}\forall a.\exists b.[[\forall x \in b.(x \in a \wedge \phi(x, \vec{an}))] \wedge [\forall x \in a.(\phi(x, \vec{an}) \to x \in b)]]$
$(\Delta_0 - Coll)$	$\forall \vec{an}.\forall a.[\forall x \in a.\exists y.\phi(x, y, \vec{an})] \to \exists b.[\forall x \in a.\exists y \in b.\phi(x, y, \vec{an})]$
$(Ad.1)$	$\forall a.Ad(a) \to trans(a)$.
$(Ad.2)$	$\forall a, b.((Ad(a) \wedge Ad(b)) \to (a \in b \vee a = b \vee b \in a))$.
$(Ad.3)$	$\forall a.(Ad(a) \to \phi^a)$*, where* ϕ *is an axiom* $(Pair)$, $(Union)$ $(\Delta_0 - Sep)$, $(\Delta_0 - Coll)$.
$(+)_{n_0}$	$\exists a, a_1, \ldots, a_{n_0}.Ad(a) \wedge (\forall x \in a.\exists c \in a.(Ad(c) \wedge x \in c)) \wedge Ad(a_1) \wedge \cdots \wedge Ad(a_{n_0}) \wedge a \in a_1 \wedge a_1 \in a_2 \wedge \cdots \wedge a_{n_0-1} \in a_{n_0}$

The arithmetical axioms are:
Some formulas $\forall\vec{n}.\exists\vec{m}.\phi(\vec{n},\vec{m})$, *where* ϕ *is quantifier free and for some primitive recursive functions* $f_1,\ldots,f_l$ ML_1W *proves* $\forall\vec{k}\in\mathbb{N}.\phi(\vec{n},f_1(\vec{k}),\ldots,f_l(\vec{k}))$. *Additionally induction:* $\phi(0)\wedge\forall n.(\phi(n)\rightarrow\phi(S(n)))\rightarrow\forall n.\phi(n)$.

4 Formalisation of the infinitary system RS

Definition 5 *We define the RS-terms and RS-formulas as follows:*

(a) $\mathcal{T}_{\mathbb{N}} := \{S^k(0) \mid k\in\mathbb{N}\}$.
$FOR_{\mathbb{N}}$ *is the set of formulas in* $KPI^+_{U\,n_0}$.

(b) $\mathcal{T}_\alpha := \{L_\alpha\}\cup$
$\{[a\in L_\alpha:\phi(a)]\cup[n\in\mathbb{N}:\psi(n)] \mid$
$\phi,\psi\in FOR_\alpha\wedge(a\in FV(\phi)\vee n\in FV(\psi))\wedge FV(\phi)\subset\{a\}\wedge FV(\psi)\subset\{n\}\}$
FOR_α *is the result of replacing in* $KPI^+_{U\,n_0}$*-formulas set terms by elements of* $\mathcal{T}_{\prec\alpha}$, *and restricting all unrestricted quantifiers to* L_α.
$K(L_\alpha):=\{\alpha\}$, $K([a\in L_\alpha:\phi(a)]\cup[n\in\mathbb{N}:\psi(n)]):=\{\alpha\}\cup K(\phi)\cup K(\psi)$.
$K(\phi):=\bigcup_{t \text{ setterm occurring in } \phi} K(t)$, $|r|:=maxK(r)$ *for* r *formula or term.*
$FOR_{\prec\alpha}:=\{\psi\mid\psi\in FOR_{|\phi|}\wedge|\phi|\prec\alpha\}$, $\mathcal{T}_\alpha:=\{t\mid t\in FOR_{|t|}\wedge|t|\prec\alpha\}$.

(c) $FOR:=\bigcup\{FOR_\alpha\mid\alpha\in OT_{n_0}\}$, $FOR_{cl}:=\{\phi\in FOR\mid FV(\phi)=\emptyset\}$,
$FOR_{cl,a}:=FOR_{cl}\cap FOR_\alpha$, $FOR_{cl,\prec\alpha}:=FOR_{cl}\cap FOR_{\prec\alpha}$.
$\mathcal{T}_{set}:=\bigcup\{\mathcal{T}_\alpha\mid\alpha\in OT_{n_0}\}$, $\mathcal{T}:=\mathcal{T}_{\mathbb{N}}\cup\mathcal{T}_{set}$, $\mathcal{T}^{0,1}:=\mathcal{T}\cup\{0,1\}$.
In the following ra,sa,ta,ra,sa,ta *denote elements of* $\mathcal{T}_{set}$, *and* r,s,t *elements of* $\mathcal{T}$.

Note, that elements of $\mathcal{T}_\alpha$, FOR_α, $\mathcal{T}_{\mathbb{N}}$, $FOR_{\mathbb{N}}$ are finite objects, therefore we can implement this easily in Martin-Löf Type Theory.

Definition 6 (a) *For* $s,t\in\mathcal{T}$ *such that* $|s|\prec|t|$ *we define* $s\mathring{\in}t$:
$s\mathring{\in}L_\alpha:=\top$, $sa\mathring{\in}[a\in L_\alpha:\phi(a)]\cup[n\in\mathbb{N}:\psi(n)]:=\phi[a:=sa]$, $sn\mathring{\in}[a\in L_\alpha:\phi(a)]\cup[n\in\mathbb{N}:\psi(n)]:=\psi[n:=sn]$.

(b) *We assign to formulas* ϕ *in* FOR_{cl} *expressions* $\phi\simeq\bigwedge_{\iota\in J}\phi_\iota$ *or* $\phi\simeq\bigvee_{\iota\in J}\phi_\iota$, *where* $J\subset\mathcal{T}^{0,1}$, *as follows:*
If $P(val(sn_1),\ldots,val(sn_k))$ *is false, then* $P(sn_1,\ldots,sn_k):\simeq\bigvee_{\iota\in\emptyset}\phi_\iota$.
$(\phi_0\vee\phi_1):=\bigvee_{\iota\in\{0,1\}}\phi_\iota$,
$sa\in ta:\simeq\bigvee_{sb\in\mathcal{T}_{|ta|}}(sb\mathring{\in}ta\wedge sa=sb)$
$sn\in sa:\simeq\bigvee_{tn\in\mathcal{T}_{\mathbb{N}}}(tn\mathring{\in}sa\wedge sn=tn)$.
$\exists n.\phi:=\simeq\bigvee_{sn\in\mathcal{T}_{\mathbb{N}}}\phi[sn]$ $\exists a\in t.\phi:\simeq\bigvee_{sa\in\mathcal{T}_{|t|}}(sa\mathring{\in}t\wedge\phi[a:=sa])$.
$Ad(s):\simeq\bigvee_{t\in J}(t=s)$ *with* $J:=\{L_\kappa|\kappa\in\mathsf{R}\wedge\kappa\preceq|s|\}$.
In all other cases, we have for some J, ψ_ι, $\neg\phi\simeq\bigvee_{\iota\in J}\psi_\iota$ *and* $\phi:\simeq\bigwedge_{\iota\in I}(\neg\psi_\iota)$.

If $\phi\simeq\bigvee_{\iota\in I}\phi_\iota$, *we call* ϕ *an* $\vee$*-formula, and if* $\phi\simeq\bigwedge_{\iota\in I}\phi_\iota$, ϕ *an* $\wedge$*-formula. In both situations let* $Index(\phi):=J$, $\phi[\iota]:=\phi_\iota$. *Note, that we can primitive recursively* *decide, whether ϕ is an $\vee$ or $\wedge$-formula, and for the J as above whether $\iota \in J$. Further $\phi[\iota]$ is primitive recursive in ϕ and ι.*

We write $\bigwedge_{\iota \in J} \phi_\iota$ for any formula ϕ such that $\phi \simeq \bigwedge_{\iota \in J} \phi_\iota$, similar for $\bigvee_{\iota \in J} \phi_\iota$.

(c) We define $rk(\theta)$ for $\theta \in FOR_{cl} \cup \mathcal{T}$ by
$rk(L_\alpha) := \omega \cdot (\alpha + 1)$,
$rk([a \in L_\alpha : \phi] \cup [n \in \mathbb{N} : \psi]) := max\{\omega \cdot \alpha + 1, rk(\phi[a := L_0]), rk(\psi[n := 0])\}$,
$rk(Ad(t)) := rk(t) + 5$,
$rk(sa \in t) := max\{rk(sa) + 6, rk(t) + 1\}$,
$rk(\exists a \in t.\phi) := max\{rk(t), rk(\phi[a := L_0]) + 2\}$,
$rk(\exists n.\phi) := rk(\phi[n := 0]) + 2$,
$rk(\phi_0 \vee \phi_1) := max\{rk(\phi_0), rk(\phi_1)\} + 1$,
$rk(\neg\phi) := rk(\phi)$ *otherwise.*

[Bu92], lemma 1.9 and definitions 1.10, 1.11, 1.12, 2.1. can be define accordingly. The α^R and $\|\Gamma\|$ are primitive recursive functions.

Lemma 7 *Assume $ML_1W \vdash B : N \to \mathcal{P}(N)$, ($\mathcal{P}(N) := N \to U$) $ML_1W \vdash \Phi : N^3 \to U$, $ML_1W \vdash \Psi : N \to U$.*
Let $\Gamma : \mathcal{P}(N) \to \mathcal{P}(N)$, $\Gamma(B) := \{k : N \mid \Psi(k) \wedge \forall l \in B(k).\exists l' \in I.\Phi(k, l, l')\}$.
Then we can define I such that $ML_1W \vdash I : \mathcal{P}(N)$, and we can prove in ML_1W: $\Gamma(I) \subset I$, and for every sub-class A of N we have $\Gamma(A) \subset A \to A \subset I$.

Proof: Define $W_\Gamma := Wk : N.\tau(k)$ with $\tau(k) := \Sigma l : N.(l \in B(k))$ (where here $l \in B(k)$ is the proposition corresponding to the property $l \in B(k)$). Let for $sup(k, s) : W_\Gamma$, $LocCor(sup(k, s)) := \Psi(k) \wedge \forall l : N.\forall p : l \in B(k).\Phi(k, l, index(s < l, p >))$.
Let $index(sup(r, s)) := r$, $pred(sup(r, s)) := \lambda x.sx$.
Define $w \prec^1_W sup(k', s) :\Leftrightarrow \exists r : \tau(k).sr = w$.
Let $w \preceq_W w' :\Leftrightarrow \exists l : N.\exists f : N \to W_\Gamma.f0 = w' \wedge fl = w \wedge \forall i < l.f(i + 1) \prec^1_W fi$.
Let for $w : W_\Gamma$, $Cor(w) :\Leftrightarrow \forall w' \preceq_W w.LocCor(w')$.
Let $I := \lambda k.\exists w : W_\Gamma.Cor(w) \wedge index(w) = k$.
Then one easily sees, that I fulfils the conditions of the theorem.

Definition 8 *(a) As in [Bu92] we define the infinitary system RS^* as the collection of all derivations generated by five inference rules:*

$$(\bigwedge)^* \quad \frac{\cdots \vdash^\rho \Gamma, \phi_\iota \cdots (\iota \in J)}{\vdash_\rho \Gamma, \bigwedge_{\iota \in J} \phi_\iota}$$

$$(\bigvee)^* \quad \frac{\vdash_\rho \Gamma, \phi_{\iota_0}, \ldots, \phi_{\iota_k}}{\vdash_\rho \Gamma, \bigvee_{\iota \in J} \phi_\iota} \quad (\textit{if } \iota_0, \ldots, \iota_k \in J \wedge K(\iota_0, \ldots, \iota_k) \subset k(\Gamma, \bigvee_{\iota \in J} \phi_\iota))$$

$$(Ad)^* \quad \frac{\cdots \vdash_\rho \Gamma, \phi[a := L_\kappa] \cdots (\kappa \preceq |t|)}{\vdash_\rho \Gamma, Ad(t) \to \phi[a := t]}, \quad \textit{if } rk(\phi[a := L_0]) \prec \rho$$

$$(Ref)^* \quad \Gamma, \phi \to \exists a \in L_\kappa.\phi^{a,\kappa}, \quad \textit{if } \phi \in \Sigma(\kappa) \wedge \kappa \in \mathsf{R} \wedge \rho \neq 0$$

$$(Found)^* \quad \Gamma, \exists a \in L_\alpha((\forall b \in a.\phi[a := b]) \wedge \not\phi), \forall a \in L_\alpha.\phi \quad \textit{if } \rho \neq 0.$$

(b) *We formalise 1 in Martin-Löf Type Theory as follows:*
In order to get unique predecessors, we replace the information on the nodes by sequences $< rule, \rho, \Gamma >$*, where* $rule =< \bigwedge, \phi >$ *or* $rule =< \bigvee, \phi, iota_0, \ldots, \iota_l >$ *or* $rule =< Ad, \phi, a, s >$ *or* $rule =< Ref, \phi, a, \kappa >$ *or* $rule =< Found, \phi, a, b, \alpha >$*.*
Then, we have

$B(<< \bigwedge, \phi >, \Gamma >) := Index(\phi)$, $\Psi_\rho(<< \bigwedge, \phi >, \Gamma >) := \phi \in \Gamma \wedge \phi \wedge -Formula$, $\Phi(< \bigwedge, \phi, \Gamma >, \iota, p) := (p)_1 = \Gamma \setminus \phi \cup \{\phi[\iota]\}$.
$B(<< \bigvee, \phi, \iota_0, \ldots, \iota_l >, \Gamma >) := Index(\phi)$, $\Psi_\rho(< \bigwedge, \phi, \iota_0, \ldots, \iota_l >, \Gamma >, p) := \phi \in \Gamma \wedge \phi \vee -Formula \wedge \iota_0, \ldots, \iota_l \in J$, $\Phi(< \bigwedge, \phi, \iota_0, \ldots, \iota_l >, \Gamma >, p) := (p)_1 = \Gamma \setminus \phi \cup \{\phi[\iota_0], \ldots, \phi[\iota_l]\}$.
The other rules are treated in a similar way.

Then with the set I_ρ *as in 7 defined for* B, Ψ_ρ, Φ, $\{(p)_1 | p \in I\}$ *is the set of sequences derivable in RS, and we define* $\vdash^*_\rho \Gamma :\Leftrightarrow \exists p \in I_\rho.(p)_1 = \Gamma$.

(c) $q \vdash^*_\rho (index(q))_1$.
$q \vdash^*_\rho \Gamma$ *is now the formalisation of, what is defined in [Bu92] definition 2.3.* $q \vdash^* \Gamma :\Leftrightarrow q \vdash^*_0 \Gamma$.

Lemmata and theorems 2.4 – 2.9 of [Bu92] follow now with nearly the same proofs. The only modifications to be made are, to define $[s \neq t]$, if either s or t is not a set-term, to add instances for the case $A = P(n_1, \ldots, n_m)$ in lemma 2.7. Further we can easily prove that for all arithmetical axioms ϕ, except the induction theorem we have $\vdash^* \phi$ (here we need, that the m_i are primitive recursive in the $\vec{n}$, so we can easily define the proof). The only case, where we really have to work is to give a cut-free proof of the induction axiom, and the reader can easily find such a proof, so for every instance ϕ of the induction axiom we have $\vdash^* \phi$.

5 $\mathcal{H}$-controlled derivations

Next step is to formalise $\mathcal{H}$-controlled derivations. However, this is only necessary for operators $\mathcal{H}_\gamma[\theta]$, where $\mathcal{H}_\gamma$ is defined in [Bu92], definition 4.3. Further, not that $\mathcal{H}_\gamma(X)$ is needed only for finite sets X. We formalise $\mathcal{H}_\gamma$ first:

Definition 9 (a) $\gamma \in C(\alpha, \beta) :\Leftrightarrow \gamma \prec \beta \vee \gamma\ \eta\ \{0, I\} \vee \exists \delta, \rho. \gamma =_{NF}' \delta + \rho \vee \gamma =_{NF} \Omega_\delta \vee (\gamma =_{NF} \psi_\delta \rho \wedge \rho \prec \gamma)$, *where* $=_{NF}$ *is defined as in definition 3.11 of [Se95].*
$C(\alpha, \beta)$ *can be defined easily as a primitive recursive set.*

(b) *For* X *being a finite subset of* N *we define* $\mathcal{H}_\gamma(X) := \{\gamma \in OT_{n_0} \mid \forall \beta, \gamma \in OT_{n_0}.(X \cap OT_{n_0} \subset C(\alpha, \beta) \wedge \gamma \prec \alpha) \rightarrow \gamma \in C(\alpha, \beta)\}$.
Note that the condition $X \cap OT_{n_0} \subset C(\alpha, \beta)$ *is primitive recursive, since* X *is finite.*

(c) $\mathcal{H}_\gamma[\theta](X) := \mathcal{H}_\gamma(k(\theta) \cup X)$.
$\alpha \in \mathcal{H}_\gamma[\theta] :\Leftrightarrow \alpha \in \mathcal{H}_\gamma[\theta](\emptyset)$.

We check easily, that for $C_\kappa(\alpha)$, as defined in [Se95], definition 3.9., we have $C_\kappa(\alpha) = C(\alpha, \psi_\kappa\alpha)$. The properties in [Bu92], lemma 4.4 b - d, 4.5 - 4.7 follow now directly from the properties of the ordinal denotation system in [Se95].

Definition 10 *(see theorem 3.8 of [Bu92]).*

Inductive definition of $\mathcal{H}_\gamma[\theta] \vdash^\alpha_\rho \Gamma$:
Assume $\{\alpha\} \subset k(\Gamma) \subset \mathcal{H}_\gamma[\theta]$. Then we can conclude $\mathcal{H}_\gamma[\theta] \vdash \Gamma$, iff one of the following cases holds:
$(\bigwedge)$ $\bigwedge_{\iota\in J} \phi_\iota \in \Gamma \wedge \forall\iota \in J.\exists\alpha_\iota \prec \alpha.(\mathcal{H}[\theta,\iota] \vdash^{\alpha_\iota}_\rho \Gamma, \phi_\iota)$
$(\bigvee)$ $\bigvee_{\iota\in J} \phi_\iota \in \Gamma \wedge \exists\iota_0 \in J.\exists\alpha_0 \prec \alpha.(\mathcal{H}[\theta] \vdash^{\alpha_0}_\rho \Gamma, \phi_{\iota_0} \wedge \iota_0 \;\eta\; J \cap (\alpha+1))$
(Cut) $rk(\psi) \prec \rho \wedge \exists\alpha_0 \prec \alpha.(\mathcal{H}[\theta] \vdash^{\alpha_0}_\rho \Gamma, \psi \wedge \mathcal{H}[\theta] \vdash^{\alpha_0}_\rho \Gamma, \neg\psi)$.
(Ref) $\exists z \in L_\kappa.\phi^{(a,\kappa)} \in \Gamma \wedge \mathcal{H}[\theta] \vdash^{\alpha_0}_\rho \Gamma, \phi \wedge \alpha_0 + 1 \prec \alpha \wedge \phi \in \Sigma(\kappa) \wedge \kappa \in \mathsf{R}$.

One sees easily, that we can formalise $\mathcal{H}$-controlled derivations in a similar way as in definition 8.
Now in [Bu92] lemma 3.9, 3.13 - 3.17 with $\mathcal{H}$ replaced by $\mathcal{H}_\gamma[\theta]$ and by omitting all conditions on $\mathcal{H}$ (we are fulfilled), and lemma 3.10, 3.11 with $\mathcal{H}$ replaced by $\mathcal{H}_\gamma$ and again by omitting conditions on $\mathcal{H}$, further lemma 4.7, theorem 4.8 and the corollary, follow with the same proofs and can be formalised in ML_1W. Theorem 3.12 reads now as follows:

Theorem 11 *For each theorem ϕ of $KPI^+_{U\,n_0}$ there exists an $m < \omega$ such that with $\lambda := \Omega_{I+m}$ for all γ $\mathcal{H}_\gamma \vdash^{\omega^{\lambda+m}}_{\lambda+m} \phi^\lambda$.*

Theorem 12 *For every arithmetical formula ϕ, if $Kpi^+_{n_0} \vdash \phi$, then $\mathcal{H}_\beta \vdash^\gamma$ for some $\gamma \prec \epsilon_{\Omega_{I+n_0}+1}$.*

Proof: Let $\lambda := \Omega_{I+n_0}$.
From $Kpi^+_{n_0} \vdash \phi$ follows by 12 $\mathcal{H}_0 \vdash^{\omega^{\lambda+m}}_{\lambda+m} \phi$, by [Bu92] 3.12 (adapted to our setting) $\mathcal{H}_0 \vdash^\alpha_{\lambda+1} \phi$ for some $\alpha \prec \epsilon_{\lambda+1}$, by [Bu92] 4.8 $\mathcal{H}_{\hat\alpha}0 \vdash^{\psi_{\Omega_1}\hat\alpha}_{\psi_{\Omega_1}\hat\alpha} \phi$ with $\hat\alpha := \omega^{\lambda+1+\alpha_0} \prec \epsilon_{\lambda+1}$, by [Bu92] 3.12 with $\gamma := \phi_{\psi_{\Omega_1}\hat\alpha}(\psi_{\Omega_1}\hat\alpha)$ $\aleph_{\hat\alpha} \vdash^\beta_0 \phi$, let $\beta := \hat\alpha$.

Lemma 13 *If $\mathcal{H}_\rho[\theta] \vdash^\alpha_\rho \Gamma, \bigwedge_{\iota\in J} \phi_\iota$, then $\mathcal{H}_\rho[\theta,\iota] \vdash^\alpha_\rho \Gamma, \phi_\iota$.*

Proof: If $\phi := \bigwedge_{\iota\in J} \phi_\iota$ is not the main formula of the last premise, the assertion follows by IH and the same rule.
Otherwise we have the case of last rule $(\bigwedge)$, $\mathcal{H}_\rho[\theta,\iota] \vdash^{\alpha_\iota}_\rho \Gamma, \phi_\iota$, or $\mathcal{H}_\rho[\theta,\iota] \vdash^{\alpha_\iota}_\rho \Gamma, \phi\phi_\iota$, in which case by IH we conclude the first case. By [Bu92] lemma 3.9 (a) follows the assertion.

6 Result

Definition 14 *We define a primitive recursive relation k rel l:*
e rel b is false, if $b \notin \Sigma^{arith}_1$.
e rel $P(S^{k_1}(0), \dots, S^{k_l}(0)) \Leftrightarrow e = 0 \wedge P(k_1, \dots, k_l)$ where on the right side stands the

primitive recursive relation corresponding to P.
$e\ rel\ \phi \wedge \psi \iff \exists l, k.e =< l, k > \wedge l\ rel\ \phi \wedge k\ rel\ \psi$.
$e\ rel\ \phi \vee \psi \iff \exists l, k.(e =< l, k > \wedge ((l = 0 \wedge krell\phi) \vee (l = 1 \wedge k\ rel\ \psi)))$.
$e\ rel\ \exists n.\phi \iff \exists l, k.e =< l, k > \wedge k\ rel\ \phi[n := S^l(0)]$.

$e\ rel\ \phi_1, \ldots, \phi_n :\Leftrightarrow e\ rel\ \phi_1 \vee \cdots \vee \phi_n$.

Lemma 15 *(a) For every formula $\phi \in \Sigma_1^{arith}$, $FV(\phi) \subset \{m_1, \ldots, m_l\}$. $ML_1W \vdash \forall k_1, \ldots, k_l.((n\ rel\ \phi[m_1 := S^{k_1}(0), \ldots, m_l := S^{k_l}(0)]) \to \widehat{\phi}[m_1 := k_1, \ldots, m_l := k_l])$, where the latter is the formula in ML_1W.*

(b) $\forall \Gamma \in \Sigma_1^{arith}.\forall \alpha, \rho, \delta.\mathcal{H}_\rho[\theta] \vdash_0^\alpha \Gamma \to \exists n.n\ rel\ \Gamma$.

Proof: b: by an easy induction on the rules. Note that only the rules ($\bigvee$) and ($\bigwedge$) occur.

Theorem 16 *Let $\phi = \forall n.\psi$, $\psi \in \Sigma_1^{arith}$. Assume $KPI^+_{U\,n_0} \vdash \phi$. Then $ML_1W \vdash \widehat{\phi}$.*

Proof: By 12 follows $\mathcal{H}_\rho \vdash_0^\alpha \phi$. Assume $k : N$. Then by 13 follows $\mathcal{H}_\rho \vdash_0^\alpha \psi[n := S^k(0)]$. Then by lemma 15 follows $\widehat{\psi}[m := k]$, therefore $\forall m.\psi$.

Corollary 17 *ML_1W proves the consistency of $KPI^+_{U\,n}$.*

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