Translating Set Theoretical Proofs into Type Theoretical Programs

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Abstract. We show how to translate proofs of Π^0_2-sentences in the fragment of set theory $\mathrm{KPI}^+_\mathrm{U}$, which is an extension of Kripke-Platek set theory, into proofs of Martin-Löf's Type Theory using a cut elimination argument. This procedure is effective. The method used is an implementation of techniques from traditional proof theoretic analysis in type theory. It follows that $\mathrm{KPI}^+_\mathrm{U}$ and $\mathrm{ML_1W}$ show the same Π^0_2-sentences and have therefore the same programs. As a side result we get that Π^0_2-sentences provable in intensional and extensional version of $\mathrm{ML_1W}$ are the same, solving partly a problem posed by M. Hofmann.

1 Introduction

Mathematics is usually developed on the basis of set theory. When trying to use type theory as a new basis for mathematics, most of mathematics has to be reformulated. This is of great use, because the step to programs is direct and one can expect to obtain good programs, provided we have an implementation of Martin-Löf's as a real programming language. However, it seems that many mathematicians will continue to work in set theory. Even when changing to type theory for the formalization, often the proofs will be developed first having classical set theory in the background.

The advantage of set theory is that it is highly flexible – e.g. there are no problems with subtyping – and that it allows to write down expressions without having to care about the type of the objects. Therefore methods for transferring directly set theoretical arguments into type theory could not only make the step from traditional mathematics to type theory easier, but as well help to program in type theory.

Π^0_2-sentences can be considered as specifications of programs and proofs of such sentences in Martin-Löf's Type Theory are implementations of such programs. In this article we will prove that all Π^0_2-sentences provable in the Kripke-Platek style set theory $\mathrm{KPI}^+_\mathrm{U}$ are theorems of $\mathrm{ML_1W}$, Martin-Löf's Type Theory with W-type and one universe as well. $\mathrm{KPI}^+_\mathrm{U}$ is **K**ripke **P**latek set theory with **u**relemente (the natural numbers), one admissible (admissibles are the recursive analogues of cardinals) closed under the step to the next admissible (a recursive **i**naccessible) and finitely many admissibles above it. The translation works for the usual variations of type theory (intensional vs. extensional, different formulations of the identity type). Since in [Se93] we have shown that all arithmetical sentences provable in $\mathrm{ML}_1\mathrm{W}$ are theorems of $\mathrm{KPI}^+_{\mathrm{U}}$ it follows that $\mathrm{KPI}^+_{\mathrm{U}}$ and $\mathrm{ML}_1\mathrm{W}$ (and the variations of $\mathrm{ML}_1\mathrm{W}$) have the same Π_2-theorems. Therefore we solve at the same time at least partly a question posed by M. Hofmann in a talk given in Munich, whether the Π^0_2-sentences of extensional and intensional type theory are the same.

It follows that transferring programs to $\mathrm{ML}_1\mathrm{W}$ from proof theoretically stronger set theories is no longer possible, therefore our result is the best possible.

Our method is heavily built on transfinite induction. In [Se97] the author has shown that $\mathrm{ML}_1\mathrm{W}$ proves transfinite induction up to the ordinals $\psi_{\Omega_1}(\Omega_{\mathrm{I}+n})$, therefore as well up to $\alpha_n := \psi_{\Omega_1}(\epsilon_{\Omega_{\mathrm{I}+n}+1}) + 1$. Transfinite induction up to α_n is exactly what we need in order to analyze $\mathrm{KPI}^+_{\mathrm{U},n}$ ($\mathrm{KPI}^+_{\mathrm{U}} = \bigcup_{n\in\omega} \mathrm{KPI}^+_{\mathrm{U},n}$). Now it is just necessary to formalize this analysis in $\mathrm{ML}_1\mathrm{W}$ using that we have transfinite induction up to α_n, and to extract the validity of Π^0_2-sentences from the cut free proofs.

We use here techniques from proof theory. These are based on the ordinal analysis of KPI as formalized by Buchholz in [Bu92]. Although it is very difficult to understand these cut elimination arguments, we think that it is not crucial to have full insight in what is going on there in order to be able to work with the techniques presented in this article.

The main problem we had to overcome for writing this article was to formalize modern proof theoretical methods (Buchholz' $\mathcal{H}$-controlled derivations), which are carried out in full set theory, in type theory. We solved this problem using proof trees with a correctness predicate.

Extensions of the result. The methods used here can be extended to all recent proof theoretical studies using infinitary derivations and ordinal analysis. Only, the type theory is not available yet in all cases, except for Mahlo universes. (For Mahlo, the author has given a formulation in [Se96]). Further, one can see easily that the well-foundedness of the W-type is not needed really here, since we have always a descent in ordinals assigned to the nodes of the tree. (For the RS^*-derivations, $||\Gamma||$ is descending). Therefore, by replacing the W-type by a recursive object obtained using the recursion theorem, which can be defined in PRA, one shows with nearly the same proof that $\mathrm{PRA} + \mathrm{TI}(\mathrm{OT}_n)$ and $\mathrm{KPI}^+_{\mathrm{U},n}$ have the same Π_2-theorems.

The restriction to *arithmetical* Π^0_2-sentences is not crucial. The quantifiers in such sentences could range as well over bigger sets like lists, free algebras or sets built using Σ and sums. We have in this article added to $\mathrm{KPI}^+_{\mathrm{U}}$ natural numbers as urelemente. By adding the types mentioned before as urelemente as well the proofs should go through without essential modifications.

Practicability It should not be too complicated to program the method used here directly in type theory, the only exception is the well ordering proof, which will be used here and seems to be too long for practical applications. However, one can think of conservative extensions of $\mathrm{ML}_1\mathrm{W}$ by adding types the elements of which represent ordinal denotations and rules for transfinite induction. Then everything shown here can be easily implemented in Martin-Löf's Type Theory.

Other approaches. Independently, W. Buchholz has taken a different approach for obtaining the same result, by using denotation systems for derivations (extending [Bu91]). He uses transfinite induction as well. His approach has the advantage of giving directly executable programs, whereas our method has the advantage of being very perspicuous and explicit.

The other approach for extracting programs from classical proofs are based on the A-translation. This can even be carried out for full set theory, as shown by Friedman (a good presentation can be found in [Be85] Sect. 1VIII.3). A lot of research is carried out for extracting practical programs using the A-translation, see for instance [BS95] or [Sch92]. However, since Martin-Löf's Type Theory is already a programming language, we believe that our approach allows to switch more easily between classical proofs and direct programming. Further, in $\mathrm{KPI}_{\mathrm{U}}^{+}$ one has constructions corresponding precisely to the different type constructors in type theory, so with our method we have good control over the strength of the methods used.

2 General Assumptions

Assumption 1. (a) We assume some coding of sequences of natural numbers. $<k_0,\dots,k_l>$ denotes the sequence $k_0,\dots,k_l$ and $(k)_i$ the i-th element (beginning with $i=0$) of the sequence k.

(b) In the following we will omit the use of Gödel-brackets.

(c) In type theory a class is an expression $\{x \mid \phi\}$ such that we can prove $x \in \mathrm{N} \Rightarrow \phi$ set, and $s\ \eta\ \{x \mid \phi\} := \phi[x := s]$. We identify primitive recursive sets with the class corresponding to it.

(d) In type theory, let $\mathcal{P}(\mathrm{N}) := \mathrm{N} \to \mathrm{U}$. For $A \in \mathcal{P}(N)$, $a \in_{\mathrm{U}} A := \mathrm{T}(A(u))$ (where $\mathrm{T}(s)$ is the set represented by an element $s \in \mathrm{U}$).

We introduce OT, the set of ordinal notations as in [Se97]:

Definition 2. (a) Let OT be defined as in [Se97], Definition 3.9. We define OT_n by:
$0, \mathrm{I}\ \eta\ \mathrm{OT}_n$.
If $\alpha,\beta\ \eta\ \mathrm{OT}_n$, $\gamma =_{\mathrm{NF}}' \alpha+\beta \vee \gamma =_{\mathrm{NF}} \phi_\alpha\beta \vee \gamma =_{\mathrm{NF}} \Omega_\gamma \vee \gamma =_{\mathrm{NF}} \psi_\beta\gamma$, $\gamma\ \eta\ \mathrm{OT} \cap \epsilon_{\Omega_{\mathrm{I}+n}+1}$, then $\gamma\ \eta\ \mathrm{OT}_n$.
In the following α,β,γ denote elements of OT_n.

(b) We restrict the ordering $\prec$ on OT to OT_n (replace $\prec$ by $\prec \cap \mathrm{OT}_n \times \mathrm{OT}_n$).

(c) Let $\mathrm{ML}_1\mathrm{W}$ be Martin-Löf's Type Theory with W-type and one universe, as for instance formulated in [Se97] (we can use other formulations of the identity type as well).

(d) For arithmetical sentences ϕ, let $\widehat{\phi}$ the canonical interpretation of ϕ in $\mathrm{ML}_1\mathrm{W}$.

Theorem 3. *If* $\mathrm{ML}_1\mathrm{W} \vdash n \in \mathrm{N} \Rightarrow \phi(n)$ type, *then* $\mathrm{ML}_1\mathrm{W} \vdash \forall k\ \eta\ \mathrm{OT}_n.((\forall l \prec k.\phi(k)) \to \phi(l)) \to \forall k\ \eta\ \mathrm{OT}_n.\phi(k)$.

Proof: Let $\mathcal{W}' := \mathcal{W}_{n+1}$ as in [Se97], Definition 5.37. Then $\mathrm{OT}_{n_0} \subset \mathcal{W}'$. Let $\psi(x) := x \;\eta\; \mathrm{OT}_n \to \phi(x)$. Then $\mathrm{Prog}(\mathcal{W}_0, (x)\psi(x))$, $\forall k \in \mathcal{W}_0.\psi(k)$ and we get the assertion.

3 The Set Theory $\mathbf{KPI}^+_{\mathbf{U},n}$

We introduce an extension of Kripke-Platek set theory $\mathrm{KPI}^+_{\mathrm{U},n}$. The best references for this theory are [Ba75] which is an excellent introduction to Kripke-Platek set theory and [Jä86], in which Kripke-Platek set theory is extended by adding a predicate for admissibles – the recursive analogue of regular cardinals – and many variations of such theories are analyzed proof theoretically. Our definition is in the spirit of [Jä86] and adds axioms, which assert the existence of one inaccessible I (which is an admissible closed under the step to the next admissible) and finitely many admissibles above it.

Definition 4. of the theory $\mathrm{KPI}^+_{\mathrm{U},n}$

(a) The language of $\mathrm{KPI}^+_{\mathrm{U},n}$ consists of infinitely many number variables, infinitely many set variables, symbols for finitely many primitive recursive relations (on the natural numbers) P of arbitrary arity (the corresponding primitive recursive relation is denoted by $\widetilde{\mathrm{P}}$), the relations Ad, $\overline{\mathrm{Ad}}$, $\in$ and $\not\in$ (the latter are written infix) and the logical symbols $\wedge, \vee, \forall, \exists$.
In the following $x^{\mathrm{nat}}, y^{\mathrm{nat}}, z^{\mathrm{nat}}, u^{\mathrm{nat}}, v^{\mathrm{nat}}$ denote number variables and x^{set}, y^{set}, ... denote set variables, to which we might add (this will apply to all future such conventions) indices or accents. $x^?, y^?, \ldots$ denote either a set or a number variable. In a formula, we will omit the superscripts nat, set and ? after the first occurrence of the variable.
We assume that $\mathrm{P}_=$ (the equality of natural numbers), $\mathrm{P}_<$ (the $<$-relation on $\mathbb{N}$) $\bot$ (the 0-ary false relation) are represented in the languge and that for every primitive recursive relation P represented in the language, the negation of this relation $\overline{\mathrm{P}}$ is represented as well.
$\top := \overline{\bot}$.

(b) Number terms are $\mathrm{S}^k(0)$ and $\mathrm{S}^k(x^{\mathrm{nat}})$, where $k \in \mathbb{N}$, $\mathrm{S}^0(r) := r$, $\mathrm{S}^{k+1}(r) := \mathrm{S}(\mathrm{S}^k(r))$. The set terms are the set variables. r^{nat}, s^{nat} and t^{nat} denote number terms. The superscript will be omitted again after the first occurrence. We define $\mathrm{val}(\mathrm{S}^k(0)) := k$.

(c) Prime formulas are $\mathrm{P}(t_1^{\mathrm{nat}}, \ldots, t_k^{\mathrm{nat}})$, where P is a symbol for a k-ary primitive recursive relation (the correspongind relation will be denoted by $\widetilde{\mathrm{P}}$), $\mathrm{Ad}(x^{\mathrm{set}})$, $\overline{\mathrm{Ad}}(x^{\mathrm{set}})$, $r^{\mathrm{set}} \in y^{\mathrm{set}}$, $r^{\mathrm{nat}} \in y^{\mathrm{set}}$, $r^{\mathrm{set}} \not\in y^{\mathrm{set}}$, $r^{\mathrm{nat}} \not\in y^{\mathrm{set}}$.

(d) Formulas are prime-formulas, and, if ϕ and ψ are formulas, then $\phi \wedge \psi$, $\phi \vee \psi$, $\forall x^?.\phi$, $\exists x^?.\phi$ are formulas, too.

(e) We define the negation of a formula by the de Morgan rules:
$\neg\mathrm{P}(t_1^{\mathrm{nat}}, \ldots, t_k^{\mathrm{nat}}) := \overline{\mathrm{P}}(t_1^{\mathrm{nat}}, \ldots, t_k^{\mathrm{nat}})$, $\neg(r \in y^{\mathrm{set}}) := r \not\in y$, $\neg\mathrm{Ad}(x^{\mathrm{set}}) := \overline{\mathrm{Ad}}(x)$, $\neg(\psi \wedge \phi) := \neg(\psi) \vee \neg(\phi)$, $\neg(\forall x^?.\phi) := \exists x^?.\neg\phi$, $\neg(\neg(\phi)) := \phi$ otherwise.
The set of free variables $\mathrm{FV}(A)$ of a formula A is defined as usual.

(f) $\phi \to \psi := \neg\phi \vee \psi$,
$\phi \leftrightarrow \psi := (\phi \to \psi) \wedge (\psi \to \phi)$.
$\forall x^? \in y^{\mathrm{set}}.\phi := \forall x^?.x \in y^{\mathrm{set}} \to \phi$,
$\exists x^? \in y^{\mathrm{set}}.\phi := \exists x^?.x \in y^{\mathrm{set}} \wedge \phi$.
$x^{\mathrm{set}} \subset y^{\mathrm{set}} := (\forall u^{\mathrm{nat}} \in x.u \in y) \wedge (\forall z^{\mathrm{set}} \in x.z \in y)$,
$x^{\mathrm{set}} = y^{\mathrm{set}} := (x \subset y \wedge y \subset x \wedge (\mathrm{Ad}(x) \leftrightarrow \mathrm{Ad}(y)))$.
$r^{\mathrm{nat}} = s^{\mathrm{nat}} := \mathrm{P}_=(r, s)$,
$r < s := \mathrm{P}_<(r, s)$.
$\mathrm{trans}(x^{\mathrm{set}}) := \forall y^{\mathrm{set}} \in x.((\forall u^{\mathrm{nat}} \in y.u \in x) \wedge (\forall z^{\mathrm{set}} \in y.z \in x))$.

(g) The quantifier $\forall x^{\mathrm{set}}$ in front of a formula ϕ is bounded, if $\forall x.\phi$ can be written as $\forall x^{\mathrm{set}} \in y^{\mathrm{set}}.\psi$, similarly for $\exists x^{\mathrm{set}}$. Quantifieres over number variables are always bounded.

(h) A formula is arithmetical, if it contains neither set terms nor set variables; it is a $\Sigma_1^{\mathrm{arith}}$-formula, if it is arithmetical and all sub-formulas starting with universal number quantifiers are of the form $\forall x^{\mathrm{nat}}.x^{\mathrm{nat}} < s^{\mathrm{nat}} \to \phi$; it is an arithmetical Π_2^0-formula, if it is of the form $\forall x^{\mathrm{nat}}.\phi$ where ϕ is a $\Sigma_1^{\mathrm{arith}}$-formula.
A formula is Δ_0, if it contains only bounded set-quantifiers. It is Σ_1, if it contains no unbounded universal set-quantifiers.
If ϕ is a formula, let $\phi^{y^{\mathrm{set}}}$ be the result of replacing in ϕ every unbounded set-quantifier $\overset{\forall}{\exists} x^{\mathrm{set}}.\psi$ by $\overset{\forall}{\exists} x^{\mathrm{set}} \in y^{\mathrm{set}}.\psi$ (note that number quantifiers remain unchanged).

(i) Γ, Δ denote multi-sets of formulas. $\Gamma, \Delta := \Gamma \cup \Delta$, $\Gamma, \phi := \Gamma \cup \{\phi\}$.

(j) The logical rules of $\mathrm{KPI}^+_{\mathrm{U},n}$ are

$$\Gamma, \phi, \neg\phi \qquad \frac{\Gamma, \phi \quad \Gamma\psi}{\Gamma\phi \wedge \psi} \qquad \frac{\Gamma, \phi}{\Gamma\phi \vee \psi} \qquad \frac{\Gamma, \psi}{\Gamma\phi \vee \psi}$$

$$\frac{\Gamma, \phi}{\Gamma, \forall x^?.\phi}\ (\text{if } x \notin \mathrm{FV}(\Gamma)) \qquad \frac{\Gamma, \phi[x^{\mathrm{nat}} := t^{\mathrm{nat}}]}{\Gamma, \exists x^{\mathrm{nat}}.\phi} \qquad \frac{\Gamma, \phi[x^{\mathrm{set}} := y^{\mathrm{set}}]}{\Gamma, \exists x^{\mathrm{set}}.\phi}$$

$$\frac{\Gamma, \phi \quad \Gamma, \neg\phi}{\Gamma}.$$

(k) Axioms of $\mathrm{KPI}^+_{\mathrm{U},n}$
$\forall \vec{z}^?$ stands in the following for $\forall z_1^?, \ldots, z_l^?$ for some l.
The set axioms are

(Ext$_1$) $\forall x^{\mathrm{set}}, y^{\mathrm{set}}, z^{\mathrm{set}}.(x = y \to x \in z \to y \in z)$,
$\forall x^{\mathrm{nat}}, y^{\mathrm{nat}}, z^{\mathrm{set}}.(x = y \to x \in z \to y \in z)$.

(Ext$_2$) $\forall x^{\mathrm{set}}.\forall y^{\mathrm{set}}.(x = y \to \mathrm{Ad}(x) \to \mathrm{Ad}(y))$.

(Found) $\forall \vec{z}^?.[(\forall x^{\mathrm{set}}.(\forall y^{\mathrm{set}} \in x^{\mathrm{set}}.\phi(y, \vec{z})) \to \phi(x, \vec{z})) \to$
$\forall x^{\mathrm{set}}.\phi(x, \vec{z})]$.

(Pair) $\forall x^?, y^?.\exists z^{\mathrm{set}}.x \in z \wedge y \in z$.

(Union) $\forall x^{\mathrm{set}}.\exists y^{\mathrm{set}}.\forall z^{\mathrm{set}} \in x.((\forall u^{\mathrm{nat}} \in z.(u \in y)) \wedge$
$(\forall v^{\mathrm{set}} \in z.(v \in y)))$.

($\Delta_0 -$ Sep) $\forall \vec{z}^?.\forall x^{\mathrm{set}}.\exists y^{\mathrm{set}}.[[\forall u^{\mathrm{nat}} \in y.\phi_{\mathrm{nat}}(u, \vec{z}))] \wedge$
$[\forall v^{\mathrm{set}} \in y.v \in x \wedge \phi_{\mathrm{set}}(v, \vec{z}))] \wedge$

$$[\forall u^{\text{nat}}.(\phi_{\text{nat}}(u,\vec{z}) \rightarrow u \in y)] \wedge$$
$$[\forall v^{\text{set}} \in x.(\phi_{\text{set}}(v,\vec{z}) \rightarrow v \in y)]]$$
where ϕ_{nat} and ϕ_{set} are Δ_0-formulas .

$(\Delta_0 - \text{Coll})$ $\forall \vec{z}^{?}.\forall x^{\text{set}}.[(\forall y^{\text{nat}} \in x.\exists v^{\text{set}}.\phi_{\text{nat}}(y,v,\vec{z})) \wedge (\forall u^{\text{set}} \in x.\exists v^{\text{set}}.\phi_{\text{set}}(u,v,\vec{z}))] \rightarrow \exists w^{\text{set}}. [(\forall y^{\text{nat}} \in x.\exists v^{\text{set}} \in w.\phi(y,v,\vec{z})) \wedge (\forall u^{\text{set}} \in x.\exists v^{\text{set}} \in w.\phi(u,v,\vec{z}))]$

where ϕ_{nat} and ϕ_{set} are Δ_0-formulas .

(Ad.1) $\forall x^{\text{set}}.\text{Ad}(x) \rightarrow (\text{trans}(x) \wedge x \neq \emptyset)$.

(Ad.2) $\forall x^{\text{set}}, y^{\text{set}}.((\text{Ad}(x) \wedge \text{Ad}(y)) \rightarrow (x \in y \vee x = y \vee y \in x))$.

(Ad.3) $\forall x^{\text{set}}.(\text{Ad}(x) \rightarrow \phi^x)$, where ϕ is an instance of an axiom (Pair), (Union), $(\Delta_0 - \text{Sep})$ or $(\Delta_0 - \text{Coll})$.

$(+)_n$ $\exists x^{\text{set}}, x_1^{\text{set}}, \ldots, x_n^{\text{set}}.\text{Ad}(x) \wedge (\forall y \in x.\exists z^{\text{set}} \in x.(\text{Ad}(z) \wedge y \in z)) \wedge \text{Ad}(x_1) \wedge \cdots \wedge \text{Ad}(x_n) \wedge x \in x_1 \wedge x_1 \in x_2 \wedge \cdots \wedge x_{n-1} \in x_n$.

The arithmetical axioms are:
Some formulas $\forall \vec{x}^{\text{nat}}.\exists \vec{y}^{\text{nat}}.\phi(\vec{x},\vec{y})$, where ϕ is quantifier free and for some primitive recursive functions $f_1, \ldots, f_l$ ML_1W proves $\forall \vec{k} \in \mathbb{N}.\hat{\phi}(\vec{n}, f_1(\vec{k}), \ldots, f_l(\vec{k}))$. Additionally induction: $\phi(0) \wedge \forall x^{\text{nat}}.(\phi(x) \rightarrow \phi(\text{S}(x))) \rightarrow \forall x^{\text{nat}}.\phi(x)$.

(l) $\text{KPI}_{\text{U}}^{+}$ is the union of all theories $\text{KPI}_{\text{U},n}^{+}$ $(n \in \omega)$.

4 Formalization of the Infinitary System RS

Let from now on be $n_0 \in \omega$ fixed.
As in [Bu92], we introduce set terms indexed by variables, where $[x^{\text{set}} \in \text{L}_\alpha : \phi(x)]$ stands for the the set $\{x \in \text{L}_\alpha \mid \phi(x)\}$, and $[x^{\text{nat}} : \phi(x)]$ for $\{x \in \mathbb{N} \mid \phi(x)\}$ (RS stands for ramified set theory):

Definition 5. We define the RS-terms and RS-formulas as follows:

(a) $\mathcal{T}_{\text{nat}} := \{\text{S}^k(0) \mid k \in \mathbb{N}\}$.

(b) We define inductively simultaneously for all $\alpha \in \text{OT}_{n_0}$ the sets of terms $\mathcal{T}'_\alpha$ and of formulas FOR'_α by:
$\text{L}_\alpha \in \mathcal{T}'_\alpha$.
If $\phi, \psi \in \text{FOR}'_\alpha$ and $x^{\text{set}} \in \text{FV}(\phi) \vee y^{\text{nat}} \in \text{FV}(\psi)$, then $[x \in \text{L}_\alpha : \phi^{\text{L}_\alpha}] \cup [y : \psi^{\text{L}_\alpha}] \in \mathcal{T}'_\alpha$.
If $r, s \in \mathcal{T}'_\alpha$, $t \in \mathcal{T}'_{\text{nat}}$, then $r \in s$, $r \notin s$, $t \in s$, $t \notin s$, $\text{Ad}(r)$, $\overline{\text{Ad}}(r) \in \text{FOR}'_\alpha$.
If $\phi, \psi \in \text{FOR}'_\alpha$, then $\phi \wedge \psi$, $\phi \vee \psi$, $\forall x^?.\phi$, $\exists x^?.\psi \in \text{FOR}'_\alpha$.

Here the free variables $\mathrm{FV}(A)$ of $A \in \mathcal{T}'_\alpha$ or $A \in \mathrm{FOR}'_\alpha$ and ϕ^r are defined as usual or before.

(c) $\mathcal{T}_\alpha := \{t \in \mathcal{T}'_\alpha \mid \mathrm{FV}(t) = \emptyset\}$, $\mathrm{FOR}_\alpha := \{A \in \mathrm{FOR}'_\alpha \mid \mathrm{FV}(A) = \emptyset\}$.
$\mathcal{T}_{\mathrm{set}} := \bigcup_{\alpha \in \mathrm{OT}_{n_0}} \mathcal{T}_\alpha$, $\mathcal{T} := \mathcal{T}_{\mathrm{set}} \cup \mathcal{T}_{\mathrm{nat}}$, $\mathrm{FOR} := \bigcup_{\alpha \in \mathrm{OT}_{n_0}} \mathrm{FOR}_\alpha$.
The elements of FOR are called RS-formulas.
An RS-formula is Δ_0, if it contains no unbounded set-quantifiers, Σ_1, if it contains no unbounded universal set-quantifiers, and $\Sigma(\alpha)$, if it is of the form $\psi^{\mathrm{L}_\alpha}[x_1^{\mathrm{set}} := t_1, \ldots, x_n^{\mathrm{set}} := t_n]$ for some Σ_1-formula ψ of $\mathrm{KPI}^+_{\mathrm{U},n_0}$ and some $t_i \in \mathcal{T}_{\prec\alpha}$. In the last situation $\phi^{t,\alpha} := \psi^t$. Further $\phi^{\beta,\alpha} := \phi^{\mathrm{L}_\beta,\alpha}$, $\phi^\beta := \phi^{\mathrm{L}_\beta}$.

(d) $\mathrm{FOR}^{\Delta_0} := \{\phi \in \mathrm{FOR} \mid \phi\ \Delta_0\text{-formula}\}$, $\mathrm{FOR}^{\Delta_0}_\alpha := \mathrm{FOR}^{\Delta_0} \cap \mathrm{FOR}_\alpha$.

(e) For $t \in \mathcal{T}$, $A \in \mathrm{FOR}$ we define $\mathrm{K}(t), \mathrm{K}(A) \subset \mathrm{OT}_{n_0}$:
$\mathrm{K}(t^{\mathrm{nat}}) := \emptyset$.
$\mathrm{K}(\mathrm{L}_\alpha) := \{\alpha\}$, $\mathrm{K}([x^{\mathrm{set}} \in \mathrm{L}_\alpha : \phi] \cup [y^{\mathrm{nat}} : \psi]) := \{\alpha\} \cup \mathrm{K}(\phi) \cup \mathrm{K}(\psi)$.
$\mathrm{K}(A)$ is the union of $\mathrm{K}(t)$ for all terms $t \in \mathcal{T}$ which occur A.
$|A| := \max \mathrm{K}(A)$ for $A \in \mathrm{FOR} \cup \mathcal{T}$.

(f) In the following $r^{\mathrm{set}}, s^{\mathrm{set}}, t^{\mathrm{set}}$ denote elements of $\mathcal{T}_{\mathrm{set}}$, and $r^?$, $s^?$, $t^?$ elements of $\mathcal{T}_{\mathrm{set}} \cup \mathcal{T}_{\mathrm{nat}}$.

Note that elements of $\mathcal{T}$ and FOR are finite objects, which can be coded easily in Martin-Löf Type Theory in such a way that all the above mentioned sets, relations and functions are primitive recursive.

In order to assign infinitary formulas to $s^? \in t^{\mathrm{set}}$ we define first the auxiliary formula $s^? \mathring{\in} t^{\mathrm{set}}$:

Definition 6. (a) For $s, t \in \mathcal{T}$ such that $|s| \prec |t|$ we define $s \mathring{\in} t$:
$s \mathring{\in} \mathrm{L}_\alpha := \top$,
$s^{\mathrm{set}} \mathring{\in} [x^{\mathrm{set}} \in \mathrm{L}_\alpha : \phi(x)] \cup [y^{\mathrm{nat}} : \psi(y)] := \phi[x := s]$,
$s^{\mathrm{nat}} \mathring{\in} [x^{\mathrm{set}} \in \mathrm{L}_\alpha : \phi(x)] \cup [y^{\mathrm{nat}} : \psi(y)] := \psi[y := s]$.

(b) We assign to formulas ϕ in FOR^{Δ_0} expressions $\phi \simeq \bigwedge_{\iota \in J} \phi_\iota$ or $\phi \simeq \bigvee_{\iota \in J} \phi_\iota$, where $J \subset \mathcal{T}$, as follows:
If $\widetilde{\mathrm{P}}(\mathrm{val}(s_1^{\mathrm{nat}}), \ldots, \mathrm{val}(s_k^{\mathrm{nat}}))$ is false then $\mathrm{P}(s_1^{\mathrm{nat}}, \ldots, s_k^{\mathrm{nat}}) :\simeq \bigvee_{\iota \in \emptyset} \phi_\iota$.
$(\phi_0 \vee \phi_1) := \bigvee_{\iota \in \{0,1\}} \phi_\iota$.
$r^{\mathrm{set}} \in s^{\mathrm{set}} :\simeq \bigvee_{t^{\mathrm{set}} \in \mathcal{T}_{\prec|s|}} (t \mathring{\in} s \wedge t = r)$.
$r^{\mathrm{nat}} \in s^{\mathrm{set}} :\simeq \bigvee_{t^{\mathrm{nat}} \in \mathcal{T}_{\mathrm{nat}}} (t \mathring{\in} s \wedge t = r)$.
$\exists x^{\mathrm{nat}}.\phi(x) :=\simeq \bigvee_{s^{\mathrm{nat}} \in \mathcal{T}_{\mathrm{nat}}} \phi(s)$.
$\exists x^{\mathrm{set}} \in t^{\mathrm{set}}.\phi :\simeq \bigvee_{s^{\mathrm{set}} \in \mathcal{T}_{\prec|t|}} (s \mathring{\in} t \wedge \phi[x := s])$.
$\mathrm{Ad}(s^{\mathrm{set}}) :\simeq \bigvee_{t^{\mathrm{set}} \in J} (t = s)$ with $J := \{\mathrm{L}_\kappa | \kappa \in \mathrm{R} \wedge \kappa \preceq |s|\}$.
In all other cases, we have for some J, ψ_ι, $\neg\phi \simeq \bigvee_{\iota \in J} \psi_\iota$ and define $\phi :\simeq \bigwedge_{\iota \in J} (\neg\psi_\iota)$.
If $\phi \simeq \bigvee_{\iota \in J} \phi_\iota$, we call ϕ an $\vee$-formula, and if $\phi \simeq \bigwedge_{\iota \in J} \phi_\iota$, ϕ an $\wedge$-formula. In both situations let $\mathrm{Index}(\phi) := J$, $\phi[\iota] := \phi_\iota$. Note that we can primitive recursively decide, whether ϕ is an $\vee$ or $\wedge$-formula, and for the J as above whether $\iota \in J$. Further $\phi[\iota]$ is primitive recursive in ϕ and ι.

We write $\bigwedge_{\iota \in J} \phi_\iota$ for any formula ϕ such that $\phi \simeq \bigwedge_{\iota \in J} \phi_\iota$, similar for $\bigvee_{\iota \in J} \phi_\iota$.

(c) We define $\mathrm{rk}(\theta)$ for $\theta \in \mathrm{FOR} \cup \mathcal{T}$ by
$\mathrm{rk}(\mathrm{L}_\alpha) := \omega \cdot (\alpha + 1)$,
$\mathrm{rk}([x^{\mathrm{set}} \in \mathrm{L}_\alpha : \phi] \cup [y^{\mathrm{nat}} : \psi]) := \max\{\omega \cdot \alpha + 1, \mathrm{rk}(\phi[x := \mathrm{L}_0]) + 2, \mathrm{rk}(\psi[y := 0]) + 2\}$,
$\mathrm{rk}(\mathrm{Ad}(t^{\mathrm{set}})) := \mathrm{rk}(t) + 5$,
$\mathrm{rk}(s^{?} \in t^{\mathrm{set}}) := \max\{\mathrm{rk}(s) + 6, \mathrm{rk}(t) + 1\}$,
$\mathrm{rk}(\exists x^{\mathrm{set}} \in t.\phi) := \max\{\mathrm{rk}(t), \mathrm{rk}(\phi[x := \mathrm{L}_0]) + 2\}$,
$\mathrm{rk}(\exists x^{\mathrm{nat}}.\phi) := \mathrm{rk}(\phi[x := 0]) + 2$,
$\mathrm{rk}(\phi_0 \vee \phi_1) := \max\{\mathrm{rk}(\phi_0), \mathrm{rk}(\phi_1)\} + 1$,
$\mathrm{rk}(\neg\phi) := \mathrm{rk}(\phi)$ otherwise.

[Bu92], Lemma 1.9 and Definitions 1.11, 1.12, 2.1. can be shown and defined accordingly. The functions $\alpha \mapsto \alpha^{\mathrm{R}}$ and $\Gamma \mapsto \|\Gamma\|$ (considered as functions on the codes) are primitive recursive.

Lemma 7. *Assume* $\mathrm{ML_1W} \vdash B \in \mathrm{N} \to \mathcal{P}(\mathrm{N})$, $\mathrm{ML_1W} \vdash \Phi \in \mathrm{N}^3 \to \mathrm{U}$, $\mathrm{ML_1W} \vdash \Psi \in \mathrm{N} \to \mathrm{U}$.
Let $\Gamma \in \mathcal{P}(\mathrm{N}) \to \mathcal{P}(\mathrm{N})$,
$\Gamma(M) := \{k \in \mathrm{N} \mid \Psi(k) \wedge \forall l \in_{\mathrm{U}} B(k).\exists l' \in_U M.\Phi(k, l, l')\}$.
(the interpretation is that $B(k)$ is the set of predecessors of a node k, $\Psi(k)$ is the condition at every nodes, and $\Phi(k, l, l')$ is the condition relating a node with its predecessors). Then we can define $\mathbb{I}$ such that $\mathrm{ML_1W} \vdash \mathbb{I} \in \mathcal{P}(\mathrm{N})$, *and we can prove in* $\mathrm{ML_1W}$*:*
$\Gamma(\mathbb{I}) \subset \mathbb{I}$, *and for every subclass A of* N *we have* $\Gamma(A) \subset A \to A \subset \mathbb{I}$.

Proof: Define $\mathrm{W}_\Gamma := \mathrm{W}k \in \mathrm{N}.\tau(k)$ with $\tau(k) := \Sigma l \in \mathrm{N}.(l \in_{\mathrm{U}} B(k))$.
Let for $\sup(k, s) \in \mathrm{W}_\Gamma$,
$\mathrm{index}(\sup(r, s)) := r$, $\mathrm{pred}(\sup(r, s), a) := s(a)$,
$\mathrm{LocCor}(\sup(k, s)) := \Psi(k) \wedge \forall l \in \mathrm{N}.\forall p \in (l \in_U B(k)).\Phi(k, l, \mathrm{index}(s(<l, p>)))$.
Define $w \prec^1_{\mathrm{W}} \sup(k', s) :\Leftrightarrow \exists r \in \tau(k).s(r) = w$.
Let $w \preceq_{\mathrm{W}} w' :\Leftrightarrow \exists l \in \mathrm{N}.\exists f \in \mathrm{N} \to \mathrm{W}_\Gamma.f(0) = w' \wedge f(l) = w \wedge \forall i < l.f(i+1) \prec^1_{\mathrm{W}} f(i)$.
Let for $w \in \mathrm{W}_\Gamma$, $\mathrm{Cor}(w) :\Leftrightarrow \forall w' \preceq_{\mathrm{W}} w.\mathrm{LocCor}(w')$.
Let $\mathbb{I} := \{k \mid \exists w \in \mathrm{W}_\Gamma.\mathrm{Cor}(w) \wedge \mathrm{index}(w) = k\}$.
Then one easily sees that $\mathbb{I}$ fulfills the conditions of the theorem.

Definition 8. (a) As in [Bu92] we define the infinitary system RS^* as the collection of all derivations generated by five inference rules ($\phi, \phi_\iota \in \mathrm{FOR}^{\Delta_0}$, $\Gamma \subset \mathrm{FOR}^{\Delta_0}$):

$$(\bigwedge)^* \quad \frac{\cdots \vdash^\rho \Gamma, \phi_\iota \cdots (\iota \in J)}{\vdash_\rho \Gamma, \bigwedge_{\iota \in J} \phi_\iota}$$

$$(\bigvee)^* \quad \frac{\vdash_\rho \Gamma, \phi_{\iota_0}, \ldots, \phi_{\iota_k}}{\vdash_\rho \Gamma, \bigvee_{\iota \in J} \phi_\iota} \quad (\text{if } \iota_0, \ldots, \iota_k \in J \text{ and}$$

$$\mathrm{K}(\iota_0,\ldots,\iota_k) \subset \mathrm{K}(\Gamma, \bigvee_{\iota\in J} \phi_\iota))^*$$

$$(\mathrm{Ad})^* \quad \frac{\cdots \vdash_\rho \Gamma, \phi[x^{\mathrm{set}} := \mathrm{L}_\kappa] \cdots (\kappa \preceq |t^{\mathrm{set}}|)}{\vdash_\rho \Gamma, \mathrm{Ad}(t) \to \phi[x := t]}, \quad \text{if } \mathrm{rk}(\phi[a := \mathrm{L}_0]) \prec \rho$$

$$(\mathrm{Ref})^* \quad \Gamma, \phi \to \exists x^{\mathrm{set}} \in \mathrm{L}_\kappa.\phi^{x,\kappa}, \quad \text{if } \phi \in \Sigma(\kappa) \wedge \kappa \in \mathrm{R} \wedge \rho \neq 0$$

$$(\mathrm{Found})^* \ \Gamma, \exists x^{\mathrm{set}} \in \mathrm{L}_\alpha((\forall y^{\mathrm{set}} \in x.\phi[z := y]) \wedge \neg\phi[z := x]),$$
$$\forall x^{\mathrm{set}} \in \mathrm{L}_\alpha.\phi[z := x] \quad \text{if } \rho \neq 0.$$

(More precisely $(\bigwedge)^*$ should be read as: if $\phi \simeq \bigwedge_{\iota\in J}\phi_\iota$, then $\dfrac{\cdots \vdash^\rho \Gamma, \phi_\iota \cdots (\iota \in J)}{\vdash_\rho \Gamma, \phi}$ is a rule etc.)

(b) We formalize (a) in Martin-Löf Type Theory as follows:
In order to get unique predecessors, we replace the information on the nodes by sequences $<rule, \rho, \Gamma>$, where $rule$ is of the form $< \bigwedge, \phi>$, $< \bigvee, \phi, \iota_0, \ldots, \iota_l>$, $<\mathrm{Ad}, \phi, x, t^{\mathrm{set}}, \kappa>$, $<\mathrm{Ref}, \phi, x, \kappa>$ or $<\mathrm{Found}, \phi, x, y, z, \alpha>$.
We want to apply Lemma 7 to B, Ψ_ρ, Φ. We define
$B(<< \bigwedge, \phi>, \Gamma>) := \mathrm{Index}(\phi)$,
$\Psi_\rho(<< \bigwedge, \phi>, \Gamma>) := \phi \in \Gamma \wedge \phi \wedge -\mathrm{Formula}$,
$\Phi(< \bigwedge, \phi, \Gamma>, \iota, p) := (((p)_1 = (\Gamma \setminus \{\phi\}) \cup \{\phi[\iota]\}) \vee ((p)_1 = \Gamma \cup \{\phi[\iota]\}))$.
$B(<< \bigvee, \phi, \iota_0, \ldots, \iota_l>, \Gamma>) := \{0\}$,
$\Psi_\rho(<< \bigvee, \phi, \iota_0, \ldots, \iota_l>, \Gamma>) := \phi \in \Gamma \wedge \phi \vee -\mathrm{Formula} \wedge$
$\iota_0, \ldots, \iota_l \in J \wedge \mathrm{K}(\iota_0, \ldots, \iota_k) \subset \mathrm{K}(\Gamma, \bigvee_{\iota\in J}, \phi)^*$,
$\Phi(<< \bigvee, \phi, \iota_0, \ldots, \iota_l>, \Gamma>, \iota, p) := (((p)_1 = (\Gamma \setminus \{\phi\}) \cup \{\phi[\iota_0], \ldots, \phi[\iota_l]\}) \vee$
$((p)_1 = \Gamma \cup \{\phi[\iota_0], \ldots, \phi[\iota_l]\}$.
The other rules are treated in a similar way.
Then with the set $\mathbb{I}_\rho$ as in Lemma 7 defined for B, Ψ_ρ, Φ, $\{(p)_1 | p \in \mathbb{I}\}$ is the set of sequences derivable in RS, and we define $\vdash^*_\rho \Gamma :\Leftrightarrow \exists p \in \mathbb{I}_\rho.(p)_1 = \Gamma$.

(c) $q \vdash^* \Gamma :\Leftrightarrow q \vdash^*_0 \Gamma$.

Lemmata and Theorems 2.4–2.9 of [Bu92] follow now with nearly the same proofs. The only modifications to be made are to define $[s^{\mathrm{nat}} \neq t^{\mathrm{nat}}]$ and to add instances for the case $A = \mathrm{P}(t_1^{\mathrm{nat}}, \ldots, t_m^{\mathrm{nat}})$ in Lemma 2.7. Further we can easily prove that for all arithmetical axioms ϕ, except the induction theorem we have $\vdash^* \phi^\lambda$. The only case, where some work is necessary is to give a cut-free proof of the induction axiom, and the reader can easily find such a proof, so for every instance ϕ of the induction axiom we have $\vdash^* \phi^\lambda$.

5 $\mathcal{H}$-controlled Derivations

The next step is to formalize $\mathcal{H}$-controlled derivations as in [Bu92]. However, this is only necessary for operators $\mathcal{H}_\gamma[\theta]$, where $\mathcal{H}_\gamma$ is defined in [Bu92], Definition 4.3. Further note that $\mathcal{H}_\gamma(X)$ is needed only for finite sets X. We formalize $\mathcal{H}_\gamma$ first:

Definition 9. (a) $\gamma \in \mathrm{C}(\alpha,\beta) :\Leftrightarrow \gamma \prec \beta \vee \gamma\ \eta\ \{0,\mathrm{I}\} \vee \exists\delta,\rho \in \mathrm{C}(\alpha,\beta).\gamma =_{\mathrm{NF}}' \delta+\rho \vee \gamma =_{\mathrm{NF}} \Omega_\delta \vee (\gamma =_{\mathrm{NF}} \psi_\delta \rho \wedge \rho \prec \alpha)$,
where $=_{\mathrm{NF}}$ is defined as in Definition 3.11 of [Se97].
$\mathrm{C}(\alpha,\beta)$ can be defined easily as a primitive recursive set.

(b) For X being a finite subset of N we define $\mathcal{H}_\gamma(X) := \{\gamma \in \mathrm{OT}_{n_0} \mid \forall\alpha,\beta \in \mathrm{OT}_{n_0}.((X \cap \mathrm{OT}_{n_0} \subset \mathrm{C}(\alpha,\beta) \wedge \gamma \prec \alpha) \to \gamma \in \mathrm{C}(\alpha,\beta))\}$.
Note that the condition $X \cap \mathrm{OT}_{n_0} \subset \mathrm{C}(\alpha,\beta)$ is primitive recursive, since X is finite.

(c) $\mathcal{H}_\gamma[\theta](X) := \mathcal{H}_\gamma(k(\theta) \cup X)$.
$\alpha \in \mathcal{H}_\gamma[\theta] :\Leftrightarrow \alpha \in \mathcal{H}_\gamma[\theta](\emptyset)$.

We check easily that for $\mathrm{C}_\kappa(\alpha)$, as defined in [Se97], Definition 3.9., we have $\mathrm{C}_\kappa(\alpha) = \mathrm{C}(\alpha,\psi_\kappa\alpha)$. The properties in [Bu92], Lemma 4.4 b–d, 4.5–4.7 follow now directly from the properties of the ordinal denotation system in [Se97].

Definition 10. (see Theorem 3.8 of [Bu92]).

Inductive definition of $\mathcal{H}_\gamma[\theta] \vdash^\alpha_\rho \Gamma$:
Assume $\{\alpha\} \subset k(\Gamma) \subset \mathcal{H}_\gamma[\theta]$. Then we can conclude $\mathcal{H}_\gamma[\theta] \vdash \Gamma$, iff one of the following cases holds:

($\bigwedge$) $\bigwedge_{\iota\in J} \phi_\iota \in \Gamma \wedge \forall\iota \in J.\exists\alpha_\iota \prec \alpha.(\mathcal{H}_\gamma[\theta,\iota] \vdash^{\alpha_\iota}_\rho \Gamma,\phi_\iota)$.

($\bigvee$) $\bigvee_{\iota\in J} \phi_\iota \in \Gamma \wedge \exists\iota_0 \in J.\exists\alpha_0 \prec \alpha.((\mathcal{H}_\gamma[\theta] \vdash^{\alpha_0}_\rho \Gamma,\phi_{\iota_0}) \wedge \iota_0 \preceq \alpha)$.

(Cut) $\mathrm{rk}(\psi) \prec \rho \wedge \exists\alpha_0 \prec \alpha.((\mathcal{H}_\gamma[\theta] \vdash^{\alpha_0}_\rho \Gamma,\psi) \wedge (\mathcal{H}_\gamma[\theta] \vdash^{\alpha_0}_\rho \Gamma,\neg\psi))$.

(Ref) $(\exists z^{\mathrm{set}} \in \mathrm{L}_\kappa.\phi^{(z,\kappa)}) \in \Gamma \wedge (\mathcal{H}_\gamma[\theta] \vdash^{\alpha_0}_\rho \Gamma,\phi) \wedge \alpha_0+1 \prec \alpha \wedge \phi \in \Sigma(\kappa) \wedge \kappa \in \mathrm{R}$.

One sees easily that we can formalize $\mathcal{H}$-controlled derivations in a similar way as in Definition 8.
Now in [Bu92] Lemma 3.9, 3.13–3.17 with $\mathcal{H}$ replaced by $\mathcal{H}_\gamma[\theta]$ and by omitting all conditions on $\mathcal{H}$ (which are fulfilled), and Lemma 3.10, 3.11 with $\mathcal{H}$ replaced by $\mathcal{H}_\gamma$ and again by omitting conditions on $\mathcal{H}$, further Lemma 4.7, Theorem 4.8 and the Corollary follow with the same proofs and can be formalized in $\mathrm{ML_1W}$.
Theorem 3.12 reads now as follows:

Theorem 11. *For every theorem ϕ of $\mathrm{KPI}^+_{\mathrm{U},n_0}$ there exists an $m<\omega$ such that with $\lambda := \Omega_{\mathrm{I}+m}$ for all γ $\mathcal{H}_\gamma \vdash^{\omega^{\lambda+m}}_{\lambda+m} \phi^\lambda$.*

Theorem 12. *For every arithmetical formula ϕ, if $\mathrm{KPI}^+_{\mathrm{U},n_0} \vdash \phi$, then $\mathcal{H}_\beta \vdash^\gamma \phi$ for some $\gamma \prec \epsilon_{\Omega_{\mathrm{I}+n_0}+1}$.*

Proof: Let $\lambda := \Omega_{\mathrm{I}+n_0}$.
From $\mathrm{KPI}^+_{\mathrm{U},n_0} \vdash \phi$ follows by Theorem 11 $\mathcal{H}_0 \vdash^{\omega^{\lambda+m}}_{\lambda+m} \phi$, by [Bu92] 3.16 (adapted to our setting) $\mathcal{H}_0 \vdash^\alpha_{\lambda+1} \phi$ for some $\alpha \prec \epsilon_{\lambda+1}$, by [Bu92] 4.8 $\mathcal{H}_{\hat\alpha} 0 \vdash^{\psi_{\Omega_1}\hat\alpha}_{\psi_{\Omega_1}\hat\alpha} \phi$ with $\hat\alpha := \omega^{\lambda+1+\alpha_0} \prec \epsilon_{\lambda+1}$, by [Bu92] 3.16 with $\gamma := \varphi_{\psi_{\Omega_1}\hat\alpha}(\psi_{\Omega_1}\hat\alpha)$ $\aleph_{\hat\alpha} \vdash^\gamma_0 \phi$, let $\beta := \hat\alpha$.

Lemma 13. *If $\mathcal{H}_\rho[\theta] \vdash^\alpha_\rho \Gamma, \bigwedge_{\iota\in J} \phi_\iota$, then $\mathcal{H}_\rho[\theta,\iota] \vdash^\alpha_\rho \Gamma,\phi_\iota$.*

Proof: If $\phi := \bigwedge_{\iota \in J} \phi_\iota$ is not the main formula of the last premise, the assertion follows by IH and the same rule.
Otherwise we have the case of last rule $(\bigwedge)$, $\mathcal{H}_\rho[\theta, \iota] \vdash_\rho^{\alpha_\iota} \Gamma, \phi_\iota$, or $\mathcal{H}_\rho[\theta, \iota] \vdash_\rho^{\alpha_\iota} \Gamma, \phi, \phi_\iota$, in which case by IH we conclude the first case. By [Bu92] Lemma 3.9 (a) follows the assertion.

6 Result

Definition 14. We define a primitive recursive relation e rel l, where l is considered as the code of a formula of RS-formula (and Gödel-brackets are omitted):
e rel l is false, if l is not the code of a Σ_1^{arith}-formula.
$e \text{ rel } \mathrm{P}(\mathrm{S}^{k_1}(0), \ldots, \mathrm{S}^{k_l}(0)) \Leftrightarrow e = 0 \wedge \widetilde{\mathrm{P}}(k_1, \ldots, k_l)$.
$e \text{ rel } \phi \wedge \psi \leftrightarrow \exists l, k. e = <l, k> \wedge l \text{ rel } \phi \wedge k \text{ rel } \psi$.
$e \text{ rel } \phi \vee \psi \leftrightarrow \exists l, k. (e = <l, k> \wedge ((l = 0 \wedge k \text{ rel } \phi) \vee (l = 1 \wedge k \text{ rel } \psi)))$.
$e \text{ rel } \exists x^{\text{nat}}. \phi \leftrightarrow \exists l, k. e = <l, k> \wedge k \text{ rel } \phi[x := \mathrm{S}^l(0)]$.
$e \text{ rel } \phi_1, \ldots, \phi_n :\Leftrightarrow e \text{ rel } (\phi_1 \vee \cdots \vee \phi_n)$.

Lemma 15. *(a) For every formula* $\phi \in \Sigma_1^{\text{arith}}$ *such that* $\mathrm{FV}(\phi) \subset \{x_1^{\text{nat}}, \ldots, x_l^{\text{nat}}\}$ *we have*
$\mathrm{ML}_1\mathrm{W} \vdash \forall k_1, \ldots, k_l \in \mathrm{N}. \forall n \in \mathrm{N}. ((n \text{ rel } \phi[x_1 := \mathrm{S}^{k_1}(0), \ldots, x_l := \mathrm{S}^{k_l}(0)])$
$\rightarrow \widehat{\phi}[x_1 := k_1, \ldots, x_l := k_l])$.
(b) $\forall \Gamma \in \Sigma_1^{\text{arith}}. \forall \alpha, \rho, \theta. \mathcal{H}_\rho[\theta] \vdash_0^\alpha \Gamma \rightarrow \exists n. n \text{ rel } \Gamma$.

Proof: b: by an easy induction on the rules. Note that only the rules $(\bigvee)$ and $(\bigwedge)$ occur.

Theorem 16. *Let* $\phi = \forall x^{\text{nat}}. \psi$, $\psi \in \Sigma_1^{\text{arith}}$. *Assume* $\mathrm{KPI}^+_{\mathrm{U}, n_0} \vdash \phi$. *Then* $\mathrm{ML}_1\mathrm{W} \vdash \widehat{\phi}$.

Proof: By Theorem 11 follows $\mathcal{H}_\rho \vdash_0^\alpha \phi$. Assume $k \in \mathrm{N}$. By Lemma 13 follows $\mathcal{H}_\rho \vdash_0^\alpha \psi[x := \mathrm{S}^k(0)]$. Then by Lemma 15 follows $\widehat{\psi}[x := k]$, therefore $\forall x. \psi$.

Corollary 17. $\mathrm{ML}_1\mathrm{W}$ *proves the consistency of* $\mathrm{KPI}^+_{\mathrm{U}, n_0}$.

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