In our thesis [5] we determined the proof theoretical strength of Martin Löf’s type theory (MLTT) with $W$-type and one universe (there was parallel work on this by E. Griffor and M. Rathjen, see [2]). One natural task is to design stronger type theories and determine its precise strength. In this talk, we will make a first, very small step in this direction and give a type theory which has strength $\|\forall \alpha \Omega_{\alpha+1} \|$ and is slightly stronger than and was motivated by the theory KPM, introduced and analyzed in [4]. We conjecture that it is stronger than the super-universe construction introduced by Palmgren in [3] and P. Dybjer’s general formulation of simultaneous inductive-recursive definitions [1]. We have heard, that E. Griffor is currently working on more general concepts, by which he will probably go far beyond what is achieved in this abstract.

The theory is an extension of (MLTT) with $W$-type and one universe (here called $V$) in the version à la Tarski, by having a rule, which assigns to every function $f$ mapping pre-universes in $V$ to pre-universes (pre-universe means, that we have just a collection of sets) a new element of $V$ which is a new universe, closed under $f$. Therefore we have the rules

\[
\begin{align*}
&f : (\Sigma x \in V . T(x) \rightarrow V) \rightarrow (\Sigma x \in V . T(x) \rightarrow V) \\
&\text{Fix}(f) : V \\
&a : T(\text{Fix}(f)) \\
&S(a, f) : V \\
\end{align*}
\]

and rules expressing that $\text{Fix}(f)$ is a sub-universe of $V$ like:

\[
\begin{align*}
&b : T(\text{Fix}(f)) \\
&x : T(S(f, b)) \Rightarrow c : T(\text{Fix}(f)) \\
&\sigma(f \in b. c : T(\text{Fix}(f)))
\end{align*}
\]
We determine a lower bound by giving a direct well-ordering proof up to the proof theoretic strength. Here we will extend methods in [5], by using the fixed-point operator to introduce universes, for finding big ordinals \( \omega_M(a) \). The upper bound for this system can be (work in progress) determined by interpreting in a similar way as in [5] the type theory in a Kripke-Platek style theory KPM*, with axioms which guarantee the existence of one recursive Mahlo \( M \) and of \( \omega \) admissibles above \( M \).


\[ \frac{b : T(\text{Fix}(f))}{x : T(S(f, b))} c : T(\text{Fix}(f)) \]
\[ S(f, \sigma_f x \in b, e) = \sigma x \in S(f, b), S(f, e) : V \]

\[
\begin{align*}
\text{We determine a lower bound by giving a direct well-ordering proof up to the proof theoretic strength. Here we will extend methods in [5], by using the fixed-point operator to introduce universes, for finding big ordinals } \omega_M(a). \text{ The upper bound for this system can be (work in progress) determined by interpreting in a similar way as in [5] the type theory in a Kripke-Platek style theory KPM*, with axioms which guarantee the existence of one recursive Mahlo } M \text{ and of } \omega \text{ admissibles above } M. \\
\end{align*}
\]

\[
\begin{align*}
\text{(i) Cons derives from the intuitive idea of consistency and, by the suggested axiomatisation, it is characterised as a non-trivial (i.e., } S \not\subseteq \text{ Cons}, \text{ hereditary (i.e., } Y \in \text{ Cons} \cap 2^X \Rightarrow X \in \text{ Cons} \text{) and regular (i.e., any } X \in \text{ Cons extends to an inclusion-maximal } Y \in \text{ Cons}) \text{ property of subsets of } S; \\
\end{align*}
\]

\[
\begin{align*}
\text{(ii) An intuitive motivation for } L_n \text{, a Lindenbaum operator, comes from the well-known Lindenbaum extension lemma. } L_n \text{ is characterised as a non-trivial, (i.e., } L_n(S) = \emptyset), \text{ extensive (i.e., } L_n(X) \subseteq 2^X), \text{ inclusive (i.e., } L_n(\emptyset) \cap 2^X \subseteq L_n(X)), \text{ and antimonotonic (i.e., } X \subseteq Y \Rightarrow L_n(Y) \subseteq L_n(X)) \text{ and regular (i.e., } X = Y \text{ for any } X \in L_n(\emptyset) \text{ and } Y \in L_n(X) \text{) operator from } P(S) \text{ to } P(P(S)); \\
\end{align*}
\]

\[
\begin{align*}
\text{(iii) Max, intuitively a property of maximality, e.g., that of deductive completeness is characterised as a non-trivial (i.e., } S \not\subseteq \text{ Max}, \text{ and regular (i.e., } X = Y \text{ for any } X \in \text{ Max and } Y \in \text{ Max} \cap 2^X) \text{ property of subsets of } S; \text{ and} \\
\end{align*}
\]

\[
\begin{align*}
\text{(iv) Sep can be motivated on intuitive grounds as logical independence. It is axiomatised here as a non-trivial (i.e., } S \not\subseteq \text{ Sep}(A), \text{ hereditary (i.e., } Y \in \text{ Sep}(A) \cap 2^X \Rightarrow X \in \text{ Sep}(A)), \text{ exclusive (i.e., } X \in \text{ Sep}(A) \implies A \not\subseteq X), \text{ convex (i.e., } X \in \text{ Sep}(A) \implies \{B : X \not\subseteq \text{ Sep}(B) \} \subseteq \text{ Sep}(A)) \text{, and regular (i.e., } X \in \text{ Sep}(A) \text{ implies that there is } Y \in \text{ Sep}(A) \cap 2^X \text{ such that } B \in Y \text{ for any } B \text{ and } C \text{ such that } Y \cup \{B \} \in \text{ Sep}(C) \text{) operator from } S \text{ to } P(P(S)). \text{ It is proved that each of these frameworks can be translated into the standard, closure-theoretic framework in a one-to-one and theorem-preserving fashion. In other words, for}
\end{align*}
\]