Extending Martin-Löf Type Theory by One Mahlo-Universe

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Abstract

We define a type theory MLM, which has proof theoretical strength slightly greater then Rathjens theory KPM. This is achieved by replacing the universe in Martin-Löf's Type Theory by a new universe V, which has the property that for every function f, mapping families of sets in V to families of sets in V, there exists a universe closed under f. We show that the proof theoretical strength of MLM is $\geq \psi_{\Omega_1}\Omega_{\mathrm{M}+\omega}$. Therefore we reach a strength slightly greater than $|\mathrm{KPM}|$ and V can be considered as a Mahlo-universe. Together with [Se96a] it follows $|\mathrm{MLM}| = \psi_{\Omega_1}(\Omega_{\mathrm{M}+\omega})$.

1 Introduction

An ordinal M is recursively Mahlo iff M is admissible and every M-recursive closed unbounded subset of M contains an admissible ordinal. Equivalently, this is the case iff M is admissible and for all Δ_0 formulas $\phi(x, y, \vec{z})$, and all $\vec{z} \in \mathrm{L}_{\mathrm{M}}$ such that $\forall x \in \mathrm{L}_{\mathrm{M}}.\exists y \in \mathrm{L}_{\mathrm{M}}.\phi(x, y, \vec{z})$ there exists an admissible ordinal $\beta < \mathrm{M}$ such that $\forall x \in \mathrm{L}_\beta \exists y \in \mathrm{L}_\beta.\phi(x, y, \vec{z})$ holds. One can easily see that M is inaccessible and that β can always be chosen to be inaccessible.

On basis of this definition Rathjen has in [Ra91] developed a formulation called KPM of Kripke Platek set theory with one recursive Mahlo ordinal and has proof theoretically analyzed it ([Ra90, Ra91, Ra94]) — a major break through in proof theory after the treatment of inaccessibles.

A universe in Martin-Löf Type Theory can be interpreted as the least fixed point of an operator, which, in the presence of the W-type, can be obtained by iterating this operator up to the first recursively inaccessible: we need one admissible to obtain closure under the inductive definition of U, but because this inductive definition refers to another inductive definition, which interprets the W-type, we need that this admissible is closed under the step to the next admissible. If we include a family of sets into the universe, the first inaccessible greater than the least α such that this family of sets is in L_α is needed. Therefore, the step to the next universe in type theory corresponds to the step to the next recursively inaccessible in set theory.

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In order to get a type-theoretic formulation of Mahloness, we replace therefore in the definition of recursive Mahlo in a first step inaccessibles by universes and Π_2-formulas by functions. We will arrive then at a universe V, which reflects functions $\mathrm{V} \to \mathrm{V}$. However, we cannot decompose elements of V, the only elimination rule for V is the Tarski-operator (or decoding function) T. Therefore reference to functions $f : \mathrm{V} \to \mathrm{V}$ only is too weak. Instead (this idea goes back to E. Palmgren, who suggested this to the author in the Oberwolfach Meeting in Mathematical Logic 1995) we have to consider functions from families of sets in V, i.e. from $\Sigma x : \mathrm{V}.\mathrm{T}(x) \to \mathrm{V}$, into itself. We arrive then at the following definition: V is a Mahlo-universe (together with Tarski-operator T), if V is a universe, and for every function $f : (\Sigma x : \mathrm{V}.\mathrm{T}(x) \to \mathrm{V}) \to (\Sigma x : \mathrm{V}.\mathrm{T}(x) \to \mathrm{V})$ there exists an element $\widehat{\mathrm{U}}_f : \mathrm{V}$, and a decoding function $\mathrm{s}_f : \mathrm{T}(\widehat{\mathrm{U}}_f) \to \mathrm{V}$ such that (with $\mathrm{U}_f := \mathrm{T}(\widehat{\mathrm{U}}_f)$) the following properties hold:

- U_f is a sub-universe of V, i.e. it is closed under ordinary type constructions. For instance, if $a : \mathrm{U}_f$ and $x : \mathrm{T}(\mathrm{s}_f(a)) \Rightarrow b : \mathrm{U}_f$, then $\widehat{\Sigma}_f x : a.b : \mathrm{U}_f$, and we have $\mathrm{s}_f(\widehat{\Sigma}_f x : a.b) = \widehat{\Sigma}_{\mathrm{V}} x : \mathrm{s}_f(a).\mathrm{s}_f(b) : \mathrm{V}$, $\mathrm{T}(\widehat{\Sigma}_{\mathrm{V}} x : a.b) = \Sigma x : \mathrm{T}(a).\mathrm{T}(b)$.
- There exists the restriction of f to U_f: $\mathrm{Res}_f : (\Sigma x : \mathrm{U}_f.\mathrm{T}(\mathrm{s}_f(x)) \to \mathrm{U}_f) \to (\Sigma x : \mathrm{U}_f.\mathrm{T}(\mathrm{s}_f(x)) \to \mathrm{U}_f)$ and with $\iota_f : (\Sigma x : \mathrm{U}_f.\mathrm{T}(\mathrm{s}_f(x)) \to \mathrm{U}_f) \to (\Sigma x : \mathrm{V}.\mathrm{T}(x) \to \mathrm{V})$, $\iota_f(\mathrm{p}(r,t)) := \mathrm{p}(\mathrm{s}_f(r), \lambda x.\mathrm{s}_f(t\ x))$ we have $\iota_f \circ \mathrm{Res}_f = f \circ \iota_f$.

In order to make it easier to work with the universe, we decompose the Σ-type and split therefore the function f into two functions $g : \mathrm{V} \to (\mathrm{T}(x) \to \mathrm{V}) \to \mathrm{V}$ and $h : \Pi x : \mathrm{V}.\Pi y : (\mathrm{T}(x) \to \mathrm{V}).\mathrm{T}(g\ x\ y) \to \mathrm{V}$. Further we define first a set $\mathrm{U}_{g,h}$ inductively and then $\mathrm{T}(\widehat{\mathrm{U}}_{g,h}) := \mathrm{U}_{g,h}$ (this was suggested by P. Martin-Löf). Note that the introduction of $\mathrm{U}_{g,h}$ is rather unproblematic and weak. That $\mathrm{U}_{g,h}$ can be represented in the form of $\widehat{\mathrm{U}}_{g,h}$ as an element of V is, what makes this theory so strong. We give the resulting theory tentatively the name MLM for extension of Martin-Löf's type theory by one Mahlo universe (note that this does not imply that this is a version of Martin-Löf's type theory — this is still a matter of discussion).

In this article we will introduce this type theory and carry out a series of well-ordering proofs, which show that the proof theoretical strength of MLM is $\geq \psi_{\Omega_1}\Omega_{\mathrm{M}+\omega}$. This is done by extending the well-ordering proofs carried out in [Se98] and showing that MLM proves transfinite induction up to $\psi_{\Omega_1}\Omega_{\mathrm{M}+m}$ for every $m \in \omega$. Since in [Se96a] we showed by modeling MLM in KPM^+ that this is an upper bound as well, we have determined the precise proof theoretical strength.

The content of this article, in which we mainly follow the proof in [Se98], is as follows: In Sect. 2 and 3 we show how to extend the ordinal notation system in [Bu93] (in which the approach of using $\mathcal{H}$-controlled derivations of [Bu92] was applied to Rathjens KPM-paper [Ra91]) in order to reach the proof theoretical strength of MLM. In Sect. 4 we introduce the type theory by modifying the version with one ordinary universe ML_{J} treated in [Se98]. In Sect. 5 we carry out the well-ordering proofs themselves. We will use most of what is done in [Se98]. We only have to carry out some technical work in order to show closure under the ψ-function (Lemma 5.2) and have to show how to reach M. The latter is mainly achieved in Lemma 5.10. Here, assuming that $a \in \mathrm{C}^a(\mathcal{W})$, $\mathrm{C}^a(\mathcal{W}) \cap a \subset \mathcal{W}$, we define a function from families of sets in V to itself by adding to a given such family one distinguished element B in the power set of N (large enough such that $a \in \mathrm{C}^a(B)$) and enlarge this family to a set of families such that, if the original family contains a distinguished set A, then $\mathrm{C}^a(A) \subset D$ for a distinguished set D in the new universe. The universe closed under this function $\mathrm{U}_{\vec{f}}$ must now necessarily have as element the distinguished set B and have the property, if $b \in \mathrm{C}^a(\mathcal{W}_{\vec{f}}) \cap a$, then, since it is closed under the function mentioned above, $b \in D$ for some distinguished set D in the power set of N in $\mathrm{U}_{\vec{f}}$, which means $b \in \mathcal{W}_{\vec{f}}$. Since $\mathcal{W}_{\vec{f}}$ is now an element of the power set in V, we obtain a distinguished set, which we can extend to a set A such that for the given $a \in \mathcal{A}^{\mathcal{W}}(\mathcal{W})$ it follows that $a \in A \subset \mathcal{W}$. From this it follows in 5.11 that, if $\psi_{\mathrm{M}}\alpha \in \mathcal{A}^{\mathcal{W}}(\mathcal{W})$, then $\psi_{\mathrm{M}}\alpha \in \mathcal{W}$, and we can add M to $\mathcal{W}$ in order to get a distinguished set containing M. $\Omega_{\mathrm{M}+n}$ is then reached by putting the W-type on top of it as in [Se98]. The general technique for carrying out well-ordering proofs goes back to Buchholz and was first presented in [Bu75] and improved in [Bu90].

Related work. E. Palmgren's studies of the super universe [Pa91] was a first major step towards the Mahlo universe. M. Rathjen, E. Griffor and E. Palmgren ([RGP98]) have defined and analyzed a substantial extension of the super universe. E. Palmgren has extended [Pa91] and defined a higher order universe ([Pa98]), which is conjectured by the author and M. Rathjen to have the strength of KPM. In [Pa98], E. Palmgren has also shown that adding elimination rules to the Mahlo universe leads to an inconsistency (in the author's opinion this only shows that we do not know yet how to formulate the correct elimination rules for it). An alternative approach to the determination of lower bounds for type theories with W-type and one universe by interpreting versions of constructive set theory was taken by E. Griffor and M. Rathjen in [GR94].

Variations of the type theory : The well-ordering proof can be carried out as well (even more easily) in the formulation à la Russell. Using the techniques from [Se93], we can easily show that our upper bound is an upper bound for the version à la Russell as well, i.e. the strength of both formulations is the same.
The well-ordering proof indicates that we do not need all functions from families of sets into itself, with $\mathcal{P}_{\mathrm{V}}(\mathrm{N}) := \mathrm{N} \to \mathrm{V}$ reflection of functions $\mathcal{P}(\mathrm{N}) \to \mathcal{P}(\mathrm{N})$ suffices.
Preliminaries 1.1 *In the following we will often refer to lemmata in [Se98], but with small changes, which we will indicate. In order to cite the lemmata shortly, we write "Lemma n.m of [Se98]" instead of "the adaption of Lemma n.m of [Se98] to the extended system".*

2 Ordinals in Set Theory

The set theoretical definition and analysis of the the ordinal functions can be found in the first part of [Bu93]. We only vary in so far as we replace M^{Γ} by $\Lambda := \min\{\alpha > \mathrm{M} | \alpha = \aleph_\alpha\} = \sup\{\alpha_n | n \in \omega\}$ with $\alpha_0 := \mathrm{M}+1$, $\alpha_{n+1} := \aleph_\alpha$, consider ordinals up to Λ, and replace SC_{M} by our definition below. With this small variation all the proofs of the first section of the [Bu93] remain the same. Unfortunately, the third section, in which the ordering between ψ-terms is determined, has to be modified.
We define $\gamma =_{\mathrm{NF}}' \delta + \rho$, $\gamma =_{\mathrm{NF}} \delta + \rho$, $\gamma =_{\mathrm{NF}} \varphi_\delta \rho$, $\gamma =_{\mathrm{NF}} \Omega_\delta$ as in [Se98], and $\gamma =_{\mathrm{NF}} \psi_\kappa \alpha :\Leftrightarrow \gamma = \psi_\kappa \alpha \wedge \kappa, \alpha \in \mathrm{C}_\kappa(\alpha)$.
Definition 2.1

$$\begin{aligned}
\mathrm{C}^0(\alpha,\beta) &:= \beta \cup \{0, \mathrm{M}\} \\
\mathrm{C}^{n+1}(\alpha,\beta) &:= \mathrm{C}^n(\alpha,\beta) \\
&\cup \{\gamma \mid \exists \delta, \rho \in \mathrm{C}^n(\alpha,\beta).\gamma =_{\mathrm{NF}}' \delta + \rho \vee \gamma =_{\mathrm{NF}} \varphi_\delta \rho \\
&\qquad \vee \gamma =_{\mathrm{NF}} \Omega_\delta \vee \gamma =_{\mathrm{NF}} \psi_\delta \rho\}
\end{aligned}$$

As usual (or as in [Se98], Lemma 1.10) we can prove $\bigcup_{n<\omega} \mathrm{C}^n(\Lambda, 0) = \mathrm{C}(\Lambda, 0)$ *and* $\bigcup_{n<\omega} \mathrm{C}^n(\alpha, \psi_\kappa \alpha) = \mathrm{C}_\kappa(\alpha)$.
Definition 2.2 *Assume* $\alpha \in \mathrm{C}(\Lambda, 0)$, $\kappa \in \mathrm{R}$.

(a) *For $\alpha \in \mathrm{C}(\Lambda,0)$ we define by recursion on $\min\{n|\alpha \in \mathrm{C}^n(\Lambda,0)\}$ the set $\mathrm{SC}_\kappa(\alpha)$:*

$$\mathrm{SC}_\kappa(\alpha) := \begin{cases} \emptyset, & \text{if } \alpha = 0, \mathrm{M}, \\ \mathrm{SC}_\kappa(\alpha_1) \cup \mathrm{SC}_\kappa(\alpha_2), & \text{if } \alpha =_{\mathrm{NF}} \alpha_1 + \alpha_2 \text{ or } \alpha =_{\mathrm{NF}} \varphi_{\alpha_1}\alpha_2 \text{ or} \\ & \quad \alpha =_{\mathrm{NF}} \Omega_{\alpha_1} \wedge \alpha_1 = \alpha_2, \\ \{\alpha\}, & \text{if } \alpha = \psi_\pi\gamma < \kappa, \\ \mathrm{SC}_\kappa(\pi) \cup \mathrm{SC}_\kappa(\gamma), & \text{if } \kappa \leq \alpha = \psi_\pi\gamma,\ \gamma, \pi \in \mathrm{C}_\pi(\gamma). \end{cases}$$

(b) *By recursion on $\min\{n|\alpha \in \mathrm{C}^n(\Lambda,0)\}$ we define $\mathrm{G}_\kappa(\alpha)$ for $\alpha \in \mathrm{C}(\Lambda,0)$:*

$$\mathrm{G}_\kappa(\alpha) := \begin{cases} \emptyset, & \text{if } \alpha = 0, \mathrm{M}, \\ \mathrm{G}_\kappa(\alpha_1) \cup \mathrm{G}_\kappa(\alpha_2), & \text{if } \alpha =_{\mathrm{NF}} \alpha_1 + \alpha_2 \text{ or } \alpha =_{\mathrm{NF}} \varphi_{\alpha_1}\alpha_2 \text{ or} \\ & \quad \alpha =_{\mathrm{NF}} \Omega_{\alpha_1} \wedge \alpha_1 = \alpha_2, \\ \emptyset, & \text{if } \alpha = \psi_\pi\gamma < \kappa, \\ \mathrm{G}_\kappa(\pi) \cup \mathrm{G}_\kappa(\gamma) \cup \{\gamma\}, & \text{if } \kappa \leq \alpha = \psi_\pi\gamma,\ \gamma, \pi \in \mathrm{C}_\pi(\gamma). \end{cases}$$

(c)

$$\mathrm{sc}_\kappa(\alpha) := \begin{cases} \mathrm{SC}_\kappa(\alpha), & \text{if } \kappa = \mathrm{M}, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Lemma 2.3 *Assume $\alpha \in \mathrm{C}(\Lambda,0)$.*

(a) $(\psi_\pi\beta = \psi_\kappa\alpha \wedge \pi, \beta \in \mathrm{C}_\pi(\beta) \wedge \alpha, \kappa \in \mathrm{C}_\kappa(\alpha)) \Rightarrow (\beta = \alpha \wedge \kappa = \pi)$.

(b) $\alpha < \kappa \Rightarrow \mathrm{G}_\kappa(\alpha) = \emptyset$.

(c) $\mathrm{SC}_\kappa(\alpha) \leq \min\{\kappa, \alpha\}$.

(d) $\alpha \in \mathrm{C}_\kappa(\beta) \Leftrightarrow (\mathrm{SC}_\kappa(\alpha) < \psi_\kappa\beta \wedge \mathrm{G}_\kappa(\alpha) < \beta)$.

(e) $(\gamma \in \mathrm{C}_\kappa(\alpha) \wedge \kappa \leq \mathrm{M}) \Rightarrow \mathrm{SC}_\mathrm{M}(\gamma) \subset \mathrm{C}_\kappa(\alpha)$.

(f) *If $\beta \in \mathrm{C}_\mathrm{M}(\beta)$, $\gamma \in \mathrm{C}_\mathrm{M}(\gamma)$, then* $\psi_\mathrm{M}\beta < \psi_\mathrm{M}\gamma \Leftrightarrow (\beta < \gamma \wedge \mathrm{SC}_\mathrm{M}(\beta) < \psi_\mathrm{M}\gamma) \vee \psi_\mathrm{M}\beta \leq \mathrm{SC}_\mathrm{M}(\gamma)$.

(g) $(\kappa, \alpha \in \mathrm{C}_\kappa(\alpha) \wedge \pi, \beta \in \mathrm{C}_\pi(\beta) \wedge \psi_\pi\beta < \kappa < \pi \wedge \mathrm{sc}_\pi(\beta) < \psi_\kappa\alpha) \Rightarrow \psi_\pi\beta < \psi_\kappa\alpha$.

Proof:
(a) Assume $\alpha \neq \beta$, w.l.o.g. $\alpha < \beta$. Then $\kappa, \alpha \in \mathrm{C}(\alpha, \psi_\kappa\alpha) \subset \mathrm{C}(\beta, \psi_\pi\beta)$, $\psi_\kappa\alpha \in \mathrm{C}_\pi(\beta) \cap \pi = \psi_\pi\beta$, a contradiction. Therefore $\mathrm{C}_\kappa(\alpha) = \mathrm{C}_\pi(\beta)$, $\kappa = \min\{\kappa' \in \mathrm{R}|\psi_\kappa\alpha < \kappa'\} = \pi$.
(b), (c): immediate.
(d): Induction on $\min\{n|\alpha \in \mathrm{C}^n(\Lambda,0)\}$.
Case $\alpha < \psi_\kappa\beta$: $\alpha \in \mathrm{C}_\kappa(\beta) \Leftrightarrow \mathrm{SC}_\kappa(\alpha) \leq \alpha < \psi_\kappa\beta \wedge \mathrm{G}_\kappa(\alpha) = \emptyset < \beta$.
Case $\psi_\kappa\beta \leq \alpha =_{\mathrm{NF}} \alpha_1 + \alpha_2, =_{\mathrm{NF}} \varphi_{\alpha_1}\alpha_2$ or $\alpha =_{\mathrm{NF}} \Omega_{\alpha_1} \wedge \alpha_1 = \alpha_2$: $\alpha \in \mathrm{C}_\kappa(\beta) \Leftrightarrow \alpha_1, \alpha_2 \in \mathrm{C}_\kappa(\beta)$, and the assertion follows by IH.
Case $\psi_\kappa\beta \leq \alpha = \psi_\pi\gamma < \kappa$: $\psi_\kappa\beta \notin \mathrm{C}_\kappa(\beta)$, $\mathrm{SC}_\kappa(\alpha) = \{\alpha\} \not< \psi_\kappa\beta$.
Case $\kappa \leq \alpha = \psi_\pi\gamma$, $\gamma, \pi \in \mathrm{C}_\pi(\gamma)$: $\alpha \in \mathrm{C}_\kappa(\beta) \Leftrightarrow (\gamma, \pi \in \mathrm{C}_\kappa(\beta) \wedge \gamma < \beta) \Leftrightarrow (\mathrm{SC}_\kappa(\pi) \cup \mathrm{SC}_\kappa(\gamma) < \psi_\kappa\beta \wedge \mathrm{G}_\kappa(\pi) \cup \mathrm{G}_\kappa(\gamma) < \beta \wedge \gamma < \beta) \Leftrightarrow \mathrm{SC}_\kappa(\alpha) < \psi_\kappa\beta \wedge \mathrm{G}_\kappa(\alpha) < \beta$.
(e): Induction on $\min\{n \in \omega \mid \gamma \in \mathrm{C}^n(\alpha, \psi_\kappa\alpha)\}$. If $\gamma = 0, \mathrm{M}$, $\gamma =_{\mathrm{NF}} \gamma_1 + \gamma_2, \varphi_{\gamma_1}\gamma_2, \Omega_{\gamma_1}$, the assertion follows directly or by IH.
If $\gamma =_{\mathrm{NF}} \psi_\pi\delta$, $\pi \leq \mathrm{M}$, it follows $\mathrm{SC}_\mathrm{M}(\gamma) = \{\gamma\} \subset \mathrm{C}_\kappa(\alpha)$.
If $\gamma =_{\mathrm{NF}} \psi_\pi\delta$, $\mathrm{M} < \pi$, it follows $\pi, \delta \in \mathrm{C}_\kappa(\alpha)$ and the assertion by IH.

(f) "$\Leftarrow$": If $\beta < \gamma \wedge \mathrm{SC_M}(\beta) < \psi_{\mathrm{M}}\gamma$, it follows $\mathrm{G_M}(\beta) < \beta < \gamma$, $\beta \in \mathrm{C_M}(\gamma)$, $\psi_{\mathrm{M}}\beta \in \mathrm{C_M}(\gamma) \cap \mathrm{M} = \psi_{\mathrm{M}}\gamma$. If $\psi_{\mathrm{M}}\beta \leq \mathrm{SC_M}(\gamma)$, follows the assertion by $\mathrm{SC_M}(\gamma) < \psi_{\mathrm{M}}\gamma$.
"$\Rightarrow$" Assume right side is wrong. Then $\mathrm{SC_M}(\gamma) < \psi_{\mathrm{M}}\beta \wedge (\gamma \leq \beta \vee \psi_{\mathrm{M}}\gamma \leq \mathrm{SC_M}(\beta))$. If $\beta = \gamma$, it follows $\psi_{\mathrm{M}}\beta = \psi_{\mathrm{M}}\gamma$, otherwise by "$\Leftarrow$" $\psi_{\mathrm{M}}\gamma < \psi_{\mathrm{M}}\beta$, contradicting the assumption.
(g): $\Omega_\pi = \pi$. If $\kappa = \Omega_{\rho+1}$, the assertion is easy. Assume $\kappa = \Omega_\kappa$. $\kappa < \mathrm{M}$, therefore $\kappa =_{\mathrm{NF}} \psi_{\mathrm{M}}\gamma$ for some $\gamma \in \mathrm{C_M}(\gamma)$. By $\kappa \in \mathrm{C}_\kappa(\alpha)$ it follows $\gamma < \alpha \wedge \gamma \in \mathrm{C}_\kappa(\alpha) \cap \mathrm{C_M}(\gamma)$. Therefore $\mathrm{SC_M}(\gamma) \subset \mathrm{C}_\kappa(\alpha) \cap \mathrm{C_M}(\gamma) \cap \mathrm{M} = \psi_\kappa\alpha$. Further $\mathrm{G_M}(\gamma) < \gamma$. $\psi_\pi\beta < \kappa = \psi_{\mathrm{M}}\gamma < \pi$. Therefore $\psi_{\mathrm{M}}\gamma \notin \mathrm{C}_\pi(\beta)$, $\beta \leq \gamma \vee \gamma \notin \mathrm{C}_\pi(\beta)$, $\beta \leq \gamma \vee \psi_\pi\beta \leq \mathrm{SC}_\pi(\gamma) \vee \beta \leq \mathrm{G}_\pi(\gamma)$.
If $\pi = \mathrm{M} \wedge \psi_\pi\beta \leq \mathrm{SC}_\pi(\gamma)$, it follows $\psi_\pi\beta \leq \mathrm{SC_M}(\gamma) \leq \psi_\kappa\alpha$, $\psi_\pi\beta \neq \psi_\kappa\alpha$ and the assertion. If $\pi = \mathrm{M} \wedge \beta \leq \gamma$, it follows, since $\psi_\pi\beta < \kappa = \psi_{\mathrm{M}}\gamma$, by (f) $(\beta < \gamma \wedge \mathrm{SC_M}(\beta) < \psi_{\mathrm{M}}\gamma) \vee \psi_{\mathrm{M}}\beta \leq \mathrm{SC_M}(\gamma)$. If $\psi_{\mathrm{M}}\beta \leq \mathrm{SC_M}(\gamma)$, it follows by $\mathrm{SC_M}(\gamma) \leq \psi_\kappa\alpha$ and $\psi_\pi\beta \neq \psi_\kappa\alpha$ the assertion. Otherwise we have $\mathrm{G_M}(\beta) < \beta < \gamma < \alpha$, $\mathrm{SC_M}(\beta) < \psi_\kappa\alpha$, $\beta \in \mathrm{C}_\kappa(\alpha)$, $\psi_\pi\beta \in \mathrm{C}_\kappa(\alpha) \cap \kappa = \psi_\kappa\alpha$.
If $\pi = \mathrm{M} \wedge \beta \leq \mathrm{G}_\pi(\gamma)$, it follows $\mathrm{G}_\pi(\gamma) \leq \gamma$, $\beta \leq \gamma$ and we get as before the assertion. The case $\pi = \mathrm{M}$ is therefore complete.
If $\pi < \mathrm{M}$, it follows from $\psi_{\mathrm{M}}\gamma < \pi < \mathrm{M}$, $\psi_\pi\beta < \psi_{\mathrm{M}}\gamma$ and by the case $\pi = \mathrm{M}$ $\psi_\pi\beta \leq \mathrm{sc_M}(\gamma) \leq \psi_\kappa\alpha$ and by $\psi_\pi\beta \neq \psi_\kappa\alpha$ therefore the assertion.
If $\mathrm{M} < \pi$, it follows $\Omega_\pi \neq \pi$.

Theorem 2.4 *Assume* $\kappa, \alpha \in \mathrm{C}_\kappa(\alpha) \wedge \pi, \beta \in \mathrm{C}_\pi(\beta)$. *Then*

$$\begin{aligned}\psi_\pi\beta < \psi_\kappa\alpha \Leftrightarrow\ & \pi \leq \psi_\kappa\alpha \\ & \vee \psi_\pi\beta \leq \mathrm{sc}_\kappa(\alpha) \\ & \vee (\pi = \kappa \wedge \beta < \alpha \wedge \mathrm{sc}_\pi(\beta) < \psi_\kappa\alpha) \\ & \vee (\psi_\pi\beta < \kappa < \pi \wedge \mathrm{sc}_\pi(\beta) < \psi_\kappa\alpha).\end{aligned}$$

Proof:
"$\Leftarrow$" If $\pi \leq \psi_\kappa\alpha$, follows the assertion by $\psi_\pi\beta < \pi$, and if $\psi_\pi\beta \leq \mathrm{sc}_\kappa(\alpha)$, by $\mathrm{sc}_\kappa(\alpha) < \psi_\kappa\alpha$. If $\pi = \kappa \wedge \beta < \alpha \wedge \mathrm{sc}_\pi(\beta) < \psi_\kappa\alpha$, it follows, if $\pi \neq \mathrm{M}$, $\psi_\pi\beta < \psi_\kappa\alpha$ as in [Bu93] 1.6(b), otherwise we have $\mathrm{SC_M}(\beta) < \psi_\kappa\alpha \wedge \mathrm{G_M}(\beta) < \beta < \alpha$, $\pi, \beta \in \mathrm{C}_\kappa(\alpha)$, therefore $\psi_\pi\beta \in \mathrm{C}_\kappa(\alpha) \cap \kappa = \psi_\kappa\alpha$. Otherwise follows the assertion by 2.3 (g).
"$\Rightarrow$" If the right side is not fulfilled, we have $\pi = \alpha \wedge \beta = \alpha$ or the right side with π, κ and α, β interchanged, therefore $\psi_\kappa\alpha \leq \psi_\pi\beta$.

3 The Notation system OT

Now we will introduce the ordinal notation system OT. We will work in Heyting-Arithmetic, which can be embedded in Martin-Löf's Type Theory in a straightforward way.

Preliminaries 3.1 *In this section, a primitive recursive set is given by a primitive recursive function* f *such that* $\forall x \in \mathbb{N}.fx = 0 \vee fx = 1$. *We write* $t \in A$ *for* $ft = 1$, *if* A *is the set denoted by* f. $A \subset B :\equiv \forall x \in \mathbb{N}.x \in A \rightarrow x \in B$ *and* $A \cong B :\equiv A \subset B \wedge B \subset A$. *In the following assume* $a, b, c, n, m, \pi, \kappa, \lambda \in \mathbb{N}$.

We will, as usual in proof theory, first introduce a system of terms and an ordering on these terms, and then define the ordinal notation system OT as a subset of these terms.

Definition 3.2 *We give an inductive definition of sets* T', Suc', P', G', Fi', R', Car' *of terms together with* length(a) *for* $a \in \mathsf{T}' \cup \mathsf{Suc}' \cup \mathsf{P}' \cup \mathsf{G}' \cup \mathsf{R}' \cup \mathsf{Fi}' \cup \mathsf{Car}'$, *where we assume some coding of the terms as natural numbers. All the sets and* length *can be defined primitive recursively.*
(T' is the set of terms, Suc' the set of successor ordinals, P' the additive principal numbers (except 0_{OT}), G' the Gamma numbers, R' the regular cardinals, Fi' the fixed points of the function $\Omega.$, I' the inaccessibles, i.e. the regular fixed points of the function $\Omega.$).

(T' *1*) $0_{\mathsf{OT}} \in \mathsf{T}'$, $\mathrm{length}(0_{\mathsf{OT}}) := 1$.

(T' *2*) *If* $n > 0$, $a_0, \ldots, a_n \in \mathsf{P}'$, *then*
$t := (a_0, \ldots, a_n) \in \mathsf{T}'$, *if* $a_n \in \mathsf{Suc}'$, *then* $t \in \mathsf{Suc}'$,
$\mathrm{length}(t) := \mathrm{length}(a_1) + \cdots \mathrm{length}(a_n)$.

(T' *3*) *If* $a, b \in \mathsf{T}'$, *then* $t := \varphi'_a b \in \mathsf{P}'$, *if* $a = b = 0_{\mathsf{OT}}$, *then* $t \in \mathsf{Suc}'$,
$\mathrm{length}(\varphi'_a b) := \mathrm{length}(a) + \mathrm{length}(b)$.
$1_{\mathsf{OT}} := \varphi'_{0_{\mathsf{OT}}} 0_{\mathsf{OT}}$.

(T' *4*) *If* $b \in \mathsf{T}'$ $\pi \in \mathsf{R}'$, *then* $t := \psi_\pi b \in \mathsf{G}'$,
and, if $\pi = \mathrm{M}$, *then* $t \in \mathsf{I}'$,
and, if $\pi \in \mathsf{I}'$, *then* $t \in \mathsf{Fi}'$,
$\mathrm{length}(t) := \mathrm{length}(\pi) + \mathrm{length}(b)$.

(T' *5*) *If* $a \in \mathsf{T}'$, *then* $t := \Omega'_a \in \mathsf{G}'$,
if $a \neq 0_{\mathsf{OT}}$, *then* $t \in \mathsf{Car}'$,
if $a \in \mathsf{Suc}'$, *then* $t \in \mathsf{R}'$,
in all cases $\mathrm{length}(t) := \mathrm{length}(a) + 1$.

(T' *6*) $\mathrm{M} \in \mathsf{I}'$, $\mathrm{length}(\mathrm{M}) := 1$.

(T' *7*) $\mathsf{I}' \subset \mathsf{R}' \subset \mathsf{G}' \subset \mathsf{P}' \subset \mathsf{T}'$, $\mathsf{I}' \subset \mathsf{Fi}' \subset \mathsf{Car}' \subset \mathsf{G}'$, $\mathsf{Suc}' \subset \mathsf{T}'$.

$\mathsf{Lim}' := \mathsf{T}' \setminus (\{0_{\mathsf{OT}}\} \cup \mathsf{Suc}')$.
For $a \in \mathsf{P}'$, $(a) := a$. $() := 0$. *So for every* $a \in \mathsf{T}'$ *there exists a unique* n *and unique* $a_1, \ldots, a_n$ *such that* $a = (a_0, \ldots, a_n)$.
After some change of the coding we assume $0 = 0_{\mathsf{OT}}$, $1 = 1_{\mathsf{OT}}$. *In the following* π, κ, λ *will indicate elements of* R', a, b, c *of* T', *whereas* n *will be used for denoting natural numbers.*

Definition 3.3 *By recursion on the length of* $\mathrm{length}(a) + \mathrm{length}(c)$ *we define simultaneously the relation* $a \prec' c$ *and, if* $c = \kappa \in \mathsf{R}'$, *finite sets* $\mathrm{SC}_\kappa(a)$, $\mathrm{sc}_\kappa(a)$ *and* $\mathrm{G}_\kappa(a)$ *such that* $\forall c \in \mathrm{sc}_\kappa(a) \cup \mathrm{G}_\kappa(a).\mathrm{length}(a) \neq 0 \to \mathrm{length}(c) < \mathrm{length}(a)$. *In this definition* $a \preceq' b$ *is an abbreviation for* $a \prec' b \vee a = b$, *and for a finite set* A $a \preceq' A$ *stands for* $\exists x \in A.a \preceq' x$, $A \prec' a$ *for* $\forall x \in A.x \prec' a$.
Later we will define $\prec$ *as the restriction of* $\prec'$ *to* OT.
$a \prec' b$ *is false, if* $a \notin \mathsf{T}' \vee b \notin \mathsf{T}' \vee a = b$.

($\prec'$ 1) $c \neq 0 \to 0 \prec' c$.

($\prec'$ 2) $n + m \geq 1$, $a_0, \ldots, a_n, b_0, \ldots, b_m \in \mathsf{P}'$, *then* $(a_0, \ldots, a_n) \prec' (b_0, \ldots, b_n) :\leftrightarrow$
$(n < m \wedge \forall i \leq n.a_i = b_i) \vee$
$(\exists j \leq \min\{n, m\}.\forall i < j.a_i = b_i \wedge a_j \prec' b_j)$.

($\prec'$ 3) *If* $a, b, c, d \in \mathsf{T}'$, *then*
$(\varphi'_a b \prec' \varphi'_c d) :\leftrightarrow$
$((a \prec' c \wedge b \prec' \varphi'_c d) \vee (a = c \wedge b \prec' d) \vee$
$(c \prec' a \wedge \varphi'_a b \preceq' d))$.

($\prec'$ 4) *If* $a, b \in \mathsf{T}'$, $c \in \mathsf{G}'$, *then*
$(\varphi'_a b \prec' c) :\leftrightarrow (\max\{a, b\} \prec' c)$.

($\prec'$ 5) $\pi, \kappa \in \mathsf{R}'$,$a, b \in \mathsf{T}'$, *then*
$(\psi_\pi a \prec' \psi_\kappa b) :\leftrightarrow$
$(\pi \preceq \psi_\kappa b \vee$
$\psi_\pi a \preceq \mathrm{sc}_\kappa(b) \vee$
$(\pi = \kappa \wedge a \prec' b \wedge \mathrm{sc}_\pi(a) \prec' \psi_\kappa b) \vee$
$(\psi_\pi a \prec' \kappa \prec' \pi \wedge \mathrm{sc}_\pi(a) \prec' \psi_\kappa b)$.

($\prec'$ 6) *If* $\pi \in \mathsf{R}' \setminus \mathsf{I}'$, $(\kappa = \Omega'_c \vee \kappa = \mathrm{M})$, $b \in \mathsf{T}'$, *then*
$(\psi_\pi b \prec' \kappa) :\leftrightarrow \pi \preceq' \kappa$.

($\prec'$ 7) $b, c \in \mathsf{T}'$, $\pi \in \mathsf{I}'$, *then*
$(\psi_\pi b \prec' \Omega'_c) :\leftrightarrow \psi_\pi b \preceq' c$.

($\prec'$ 8) *If* $b \in \mathsf{T}'$, $\pi \in \mathsf{I}'$, *then*

$\psi_\pi b \prec' \mathrm{M}$.

($\prec'$ 9) *If $a, c \in \mathsf{T}'$, then*
$(\Omega'_a \prec' \Omega'_c) :\leftrightarrow (a \prec' c)$.

($\prec'$ 10) *If $a \in \mathsf{T}'$, then*
$(\Omega'_a \prec' \mathrm{M}) :\leftrightarrow (a \prec' \mathrm{M})$.

($\prec'$ 11) *In all other cases $a \prec' b :\leftrightarrow \neg(b \preceq' a)$.*

$$\mathrm{SC}_\kappa(a) := \begin{cases} \emptyset, & \text{if } a = 0, \mathrm{M}, \\ \mathrm{SC}_\kappa(a_1) \cup \cdots \cup \mathrm{SC}_\kappa(a_n), & \text{if } a = (a_1, \ldots, a_n),\ n \geq 2, \\ \mathrm{SC}_\kappa(a_1) \cup \mathrm{SC}_\kappa(a_2), & \text{if } a = \varphi'_{a_1} a_2 \vee (a = \Omega'_{a_1} \wedge a_1 = a_2), \\ \{a\} & a = \psi_\lambda c \prec' \kappa, \\ \mathrm{SC}_\kappa(\lambda) \cup \mathrm{SC}_\kappa(c), & \text{if } \kappa \preceq' a = \psi_\lambda c. \end{cases}$$

$$\mathrm{sc}_\kappa(a) := \begin{cases} \mathrm{SC}_{\mathrm{M}}(a), & \text{if } \kappa = \mathrm{M}, \\ \emptyset & \text{otherwise.} \end{cases}$$

$$\mathrm{G}_\kappa(a) := \begin{cases} \emptyset, & \text{if } a = 0, \mathrm{M}, \\ \mathrm{G}_\kappa(a_1) \cup \cdots \cup \mathrm{G}_\kappa(a_n), & \text{if } a = (a_1, \ldots, a_n),\ n \geq 2, \\ \mathrm{G}_\kappa(a_1) \cup \mathrm{G}_\kappa(a_2), & \text{if } a = \varphi'_{a_1} a_2 \vee (a = \Omega'_{a_1} \wedge a_1 = a_2), \\ \emptyset, & \text{if } a = \psi_\lambda c \prec' \kappa, \\ \mathrm{G}_\kappa(\lambda) \cup \mathrm{G}_\kappa(c) \cup \{c\}, & \text{if } \kappa \prec' a = \psi_\lambda c. \end{cases}$$

We define for finite sets M, M' $M \preceq' M' :\Leftrightarrow \forall x \in M \exists y \in M'(x \preceq' y)$, $M \prec' M' :\Leftrightarrow \forall x \in M \exists y \in M'(x \prec' y)$.

Lemma 3.4 *$(\prec')$ is a linear ordering on T'.*

Proof: easy, but very long.

Definition 3.5 *(a) We define the (primitive-recursively decidable) subset $Cr'(a)$ of T':*
$0, (a_1, \ldots, a_n) \notin Cr'(a)$.
$\varphi'_b c \in Cr'(a) :\Leftrightarrow a \prec' b$.
If $b \in \mathsf{G}'$, then $b \in Cr'(a) :\Leftrightarrow a \prec b$.

(b) We define $\widetilde{\mathrm{C}}_a(b)$ as the primitive recursively definable set such that
$c \in \widetilde{\mathrm{C}}_a(b) :\Leftrightarrow c \in \mathsf{T}' \wedge \mathrm{SC}_a(c) \prec \psi_a b \wedge \mathrm{G}_a(c) \prec b$.

Definition 3.6 *We define the set OT of ordinal notations, which will be a subset of T', by:*

(OT *1*) $0 \in \mathsf{OT}$.

(OT *2*) *If $n > 0$, $a_0, \ldots, a_n \in \mathsf{OT} \cap \mathsf{P}'$, $a_n \preceq \cdots \preceq a_0$, then*
$(a_0, \ldots, a_n) \in \mathsf{OT}$,

(OT *3*) *If $a, b \in \mathsf{OT}$, $b \notin Cr'(a)$, $\neg(b = 0 \wedge a \in \mathsf{G}')$, then*
$\varphi'_a b \in \mathsf{OT}$.

(OT *4*) *If $b \in \mathsf{OT}$ $\pi \in \mathsf{R}' \cap \mathsf{OT}$, $b, \pi \in \widetilde{\mathrm{C}}_\pi(b)$, then*
$\psi_\pi b \in \mathsf{OT}$,

(OT *5*) *If $a \in \mathsf{OT} \setminus (\mathsf{Fi}' \cup \{0\})$, then $\Omega'_a \in \mathsf{OT}$.*

(OT *6*) $\mathrm{M} \in \mathsf{OT}$.

$\mathsf{I} := \mathsf{I}' \cap \mathsf{OT}$, $\mathsf{Fi} := \mathsf{Fi}' \cap \mathsf{OT}$, $\mathsf{R} := \mathsf{R}' \cap \mathsf{OT}$, $\mathsf{G} := \mathsf{G}' \cap \mathsf{OT}$, $\mathsf{P} := \mathsf{P}' \cap \mathsf{OT}$, $\mathsf{Suc} := \mathsf{Suc}' \cap \mathsf{OT}$, $\mathsf{Car} := \mathsf{Car}' \cap \mathsf{OT}$, $\mathrm{Cr}(a) := \mathrm{Cr}'(a) \cap \mathsf{OT}$, $\mathrm{C}_a(b) := \widetilde{\mathrm{C}}_a(b) \cap \mathsf{OT}$.
$a \prec b :\Leftrightarrow a \prec' b \wedge a \in \mathsf{OT} \wedge b \in \mathsf{OT}$, $a \preceq b :\Leftrightarrow a \preceq' b \wedge a \in \mathsf{OT} \wedge b \in \mathsf{OT}$.
In the following, we write sometimes a for the primitive recursively decidable set $\{x \in \mathsf{OT} | x \prec a\}$.

We define all the functions in OT as in Sect. 2.2 of [Se98] and have the same lemmata, except that we don't define a^{+Fi}, a^{-Fi}, $a^{+\mathrm{I}}$, $a^{-\mathrm{I}}$, replace in some lemmata I by M and define $\tilde{a}$, a^{+}, κ^{-} as follows:

Definition 3.7 *(a) For $a \in \mathsf{T}'$, we define $\tilde{a}$, a^{+}, ($\tilde{a}$ will be the largest cardinal below, a^{+} the least element of Car greater than a).*

$\tilde{0} := 0$, $0^{+} := \Omega_1$.

If $a = (a_0, \dots, a_n)$, $n > 0$, then $\tilde{a} := \tilde{a}_0$, $a^{+} := a_0^{+}$.

If $a =_{\mathrm{NF}} \varphi_b c$, then with $d := \max\{b, c\}$ we define $\tilde{a} := \tilde{d}$, $a^{+} := d^{+}$.

If $a = \psi_b c$, $b \notin \mathsf{I}$, then $\tilde{a} := b^{-}$, $a^{+} := b$.

If $a = \psi_b c$, $b \in \mathsf{I}$, $\tilde{a} := a$, $a^{+} := \Omega_{a+1}$.

If $a =_{\mathrm{NF}} \Omega_b$, $\tilde{a} := a$, $a^{+} := \Omega_{b+1}$.

$\tilde{\mathrm{M}} := \mathrm{M}$, $\mathrm{M}^{+} := \Omega_{\mathrm{M}+1}$.

(b) For $\kappa \in \mathsf{R}$, $\kappa^{-} := \begin{cases} \Omega_s, & \text{if } \kappa = \Omega_{s+1}, \\ \kappa, & \text{if } \kappa \in \mathsf{I}. \end{cases}$

Lemma 3.8 *Assume $a \in \mathsf{OT}$, $\kappa, \pi \in \mathsf{R}$.*

(a) $\mathrm{SC}_\kappa(a) \preceq \min\{\kappa, a\}$.

(b) $a \prec \kappa \Rightarrow \mathrm{G}_\kappa(a) = \emptyset$.

(c) $\mathrm{SC}_\pi(a) \prec \kappa \prec \pi \Rightarrow \mathrm{SC}_\kappa(a) = \mathrm{SC}_\pi(a)$.

Proof: (a), (b): easy.
(c) by induction on length(a): If $a = 0, \mathrm{M}$ or $a =_{\mathrm{NF}} a_0 + a_1, \varphi_{a_0} a_1, \Omega_{a_0}$, this is obvious or follows by IH. If $a = \psi_{\kappa'} e$, it follows, if $a \prec \pi$, $\{a\} \cong \mathrm{SC}_\pi(a) \prec \kappa$, $\mathrm{SC}_\kappa(a) \cong \{a\} \cong \mathrm{SC}_\pi(a)$, and if $\pi \preceq a$, it follows by IH $\mathrm{SC}_\kappa(a) = \mathrm{SC}_\kappa(\kappa') \cup \mathrm{SC}_\kappa(e) = \mathrm{SC}_\pi(a)$ by IH.

4 Definition of the Type Theory MLM

Definition 4.1 *(a) The symbols of* MLM *are infinitely many variables z_i ($i \in \omega$); the symbols $\Rightarrow$, :, ,, (,), =, λ,* type, context*; the term constructors (with their arity in parenthesis)* 0 *(0),* l_k *(for each $l < k$, with arity 0),* S *(1),* i *(1),* j *(1),* p_0 *(1),* p_1 *(1),* **r** *(1),* λ *(1),* E *(2),* p *(2),* sup *(2),* R *(2),* Ap *(2),* J *(2),* D *(3),* P *(3), and* C_n *($n \in \omega$, arity $n+1$); the* U*-universe-term-constructors, which are term-constructors as well:* $\widehat{\mathrm{N}}$ *(2),* $\widehat{\mathrm{N}}_k$ *(for each $k \in \omega$, with arity 2),* $\widehat{+}$ *(4),*$\widehat{\Pi}$ *(4),* $\widehat{\Sigma}$ *(4),* $\widehat{\mathrm{W}}$ *(4) and* $\widehat{\mathrm{I}}$ *(5); the* V*-universe-term-constructors, which are again term-constructors:* $\widehat{\mathrm{N}}_{\mathrm{V}}$ *(0),* $\widehat{\mathrm{N}}_{\mathrm{V},k}$ *(for each $k \in \omega$, with arity 0),* $\widehat{+}_{\mathrm{V}}$ *(2),* $\widehat{\Pi}_{\mathrm{V}}$ *(2),* $\widehat{\Sigma}_{\mathrm{V}}$ *(2),* $\widehat{\mathrm{W}}_{\mathrm{V}}$ *(2) and* $\widehat{\mathrm{I}}_{\mathrm{V}}$ *(3) (which correspond to the* U*-universe-term-constructors* $\widehat{\mathrm{N}}$, $\widehat{\mathrm{N}}_k$, $\widehat{\Pi}$, $\widehat{\Sigma}$, $\widehat{\mathrm{W}}$, $\widehat{\mathrm{I}}$ *respectively); the Mahlo-term-constructors which are term-constructors* $\widehat{\mathrm{U}}$ *(2),* s *(3),* Res0 *(4),* Res1 *(5); the type constructors with their arity* N_k *(for each $k \in \omega$, arity 0),* N *(0),* + *(2),* Π *(2),* Σ *(2),* W *(2),* I *(3),* V *(0),* T *(1) and* U *(2).*

(b) The b-objects are variables, further $(x_1, \dots, x_n)b$ and $C(b_1, \dots, b_n)$, if C is an n-ary term- or type-constructor, $x_1, \dots, x_n$ are variables and $b, b_1, \dots, b_n$ are b-objects. The set $\mathrm{FV}(b)$ *of free variables of a b-object b are defined in the usual way. We write* +, $\widehat{+}_{\mathrm{V}}$ *infix (for instance $(a + b)$ for $+(a, b)$), $a \widehat{+}_{f,g} b$ for $\widehat{+}(f, g, a, b)$, $\lambda x.t$ for $\lambda((x)t)$, for $E \in \{\Sigma, \Pi, \mathrm{W}, \widehat{\Sigma}_{\mathrm{V}}, \widehat{\Pi}_{\mathrm{V}}, \widehat{\mathrm{W}}_{\mathrm{V}}\}$ $Ex : s.t := E(s, (x)t)$, for $E \in \{\widehat{\Pi}, \widehat{\mathrm{W}}, \widehat{\Sigma}\}$ $E_{f,g} x : s.t := E(f, g, s, (x)t)$, $\mathrm{s}_{f,g}(a) := s(f, g, a)$, $\mathrm{Res0}_{f,g}(r, s) := \mathrm{Res0}(f, g, r, s)$, similarly for $\widehat{\mathrm{N}}$, $\widehat{\mathrm{N}}_k$, Res1, U, $\widehat{\mathrm{U}}$, $\widehat{\mathrm{I}}$. Further we write (rs) for $\mathrm{Ap}(r, s)$.*
We have the usual conventions about omitting brackets, especially the scope of λx.

is as long as possible, for instance $\lambda x.st$ *should be read as* $\lambda x.(st)$*. We define for b-objects* $b_1, \dots, b_n, b$ *and variables* $x_1, \dots, x_n$ *the simultaneous substitution* $b[x_1 := b_1, \dots, x_n := b_n]$ *(using the convention that, if* $x_i = x_j$*, then the first substitution applies) and "*$b[x_1 := b_1, \dots, x_n := b_n]$ *is an allowed substitution" in the usual way.* α*-equality* $(=_\alpha)$ *is defined again as always.*

(c) *The set of m-terms (for* MLM*-objects) is inductively defined as: a variable* x *is an m-term; if* $l < k$, $l, k \in \mathbb{N}$*, then* l_k *is an m-term; if* r, s, t, s_0, t_0 *are m-terms,* $x, y, z, x', y', z' \in \mathrm{Var_{ML}}$, $x \neq y \neq z \neq x$, $x' \neq y' \neq z' \neq y'$, $f := (x', y')s_0$, $g := (x', y', z')t_0$ *and* $\vec{f}$ *abbreviates* f, g*, then* 0, $S(r)$, $\lambda x.r$, $\mathrm{p}(r, s)$, $\mathrm{i}(r)$, $\mathrm{j}(r)$, $\sup(r, s)$, $\mathbf{r}(r)$, $\mathrm{P}(r, s, (x, y)t)$, $\mathrm{Ap}(r, s)$, $\mathrm{p}_0(r)$, $\mathrm{p}_1(r)$, $\mathrm{R}(r, (x, y, z)s)$, $\mathrm{D}(r, (x)s, (y)t)$, $\mathrm{J}(r, (x)s)$; $\widehat{\Pi}_{\vec{f}} x : r.s$, $\widehat{\Sigma}_{\vec{f}} x : r.s$, $\widehat{\mathrm{W}}_{\vec{f}} x : r.s$, $r \widehat{+}_{\vec{f}} s$, $\widehat{\mathrm{I}}_{\vec{f}}(r, s, t)$, $\widehat{\mathrm{N}}_{\vec{f}}$, $\widehat{\mathrm{N}}_{k,\vec{f}}$ *(for* $k \in \omega$*)*, $\widehat{\mathrm{U}}_{\vec{f}}$, $\mathrm{s}_{\vec{f}}(r)$, $\mathrm{Res0}_{\vec{f}}(r, (x)s)$, $\mathrm{Res1}_{\vec{f}}(r, (x)s, t)$; $\widehat{\mathrm{N}}_{\mathrm{V}}$, $\widehat{\mathrm{N}}_{k,\mathrm{V}}$ *(for* $k \in \omega$*)*, $\widehat{\Pi}_{\mathrm{V}} x : r.s$, $\widehat{\Sigma}_{\mathrm{V}} x : r.s$, $\widehat{\mathrm{W}}_{\mathrm{V}} x : r.s$, $r \widehat{+}_{\mathrm{V}} s$, $\widehat{\mathrm{I}}_{\mathrm{V}}(r, s, t)$ *are m-terms; if* $n \in \mathbb{N}$ *and* $r, s_1, \dots, s_n$ *are m-terms, then* $\mathrm{C}_n(r, s_1, \dots, s_n)$ *is an m-term.*
Let $\mathrm{Term_{Cl}}$ *be the set of closed m-terms.*

(d) *The pseudo-term-constructors are* $\widehat{\mathrm{N}}_{\vec{f}}$, $\widehat{\mathrm{N}}_{k,\vec{f}}$ $(k \in \omega)$, $\widehat{\Pi}_{\vec{f}}$, $\widehat{\Sigma}_{\vec{f}}$, $\widehat{\mathrm{W}}_{\vec{f}}$, $\widehat{+}_{\vec{f}}$, $\widehat{\mathrm{I}}_{\vec{f}}$ *for any m-term* f*. If* $C \in \{\widehat{\mathrm{N}}, \widehat{\mathrm{N}}_k, \widehat{\Pi}, \widehat{\Sigma}, \widehat{\mathrm{W}}, \widehat{+}, \widehat{\mathrm{I}}\}$*,* C *of arity* n*,* $C_{\vec{f}}$ *the corresponding pseudo-term-constructor, then* $C_{\vec{f}}$ *can be treated as a constructor of arity* $n - 2$ *by* $C_{\vec{f}}(t_1, \dots, t_{n-1}) := C(f, g, t_1, \dots, t_{n-1})$*.* C_{V} *is the* V*-universe-term-constructor, corresponding to* C*.*

(e) *Further we define* $\mathrm{T_V} := \mathrm{T}$, $\mathrm{T}_{\vec{f}}(t) := \mathrm{T}(\mathrm{s}_{\vec{f}}(t))$. $\mathrm{U_V} := \mathrm{V}$.

(f) *The m-types are* N_k $(k \in \omega)$, N, V*; and if* A, B *are m-types,* $x \in \mathrm{Var_{ML}}$*,* r, s *m-terms,* $\vec{f}$ *as above, then* $\mathrm{T}(r)$, $\Pi x : A.B$, $\Sigma x : A.B$, $\mathrm{W} x : A.B$, $A + B$, $\mathrm{U}_{\vec{f}}$ $\mathrm{I}(A, r, s)$ *are m-types.*

(g) *An m-context-piece is a string* $x_1 : A_1, \dots, x_n : A_n$*, where* $n \geq 0$*,* x_i *different variables,* A_i *m-types.*
An m-context is an m-context-piece $x_1 : A_1, \dots, x_n : A_n$ *such that* $\mathrm{FV}(A_i) \subset \{x_1, \dots, x_{i-1}\}$ *for* $i = 1, \dots, n$*.*
A m-judgements are context, $A = B$*, and* $s = t : A$ *where* A*,* B *are m-types,* s*,* t *are m-terms.*
We treat the usual judgements A type *and* $s : A$ *as abbreviations:* $A \text{ type} :\equiv A = A$*,* $s : A :\equiv s = s : A$*.*

(h) *We abbreviate* $[\vec{x} := \vec{t}] := [x_1 := t_1, \dots, x_n := t_n]$*, if* $\vec{x} = x_1, \dots, x_n$ *and* $\vec{t} = t_1, \dots, t_n$*.*
$[x_1 := t_1, \dots, x_n := t_n] \setminus \{y\} := [\vec{x} := \vec{t}] \setminus \{y\}$ *is the result of omitting in* $[\vec{x} := \vec{t}]$ *all* $x_i := t_i$ *such that* $x_i = y$*, and* $[\vec{x} := \vec{t}] \setminus \{y_1, \dots, y_m\} := (\cdots(([\vec{x} := \vec{t}] \setminus \{y_1\}) \setminus \{y_2\}) \cdots \setminus \{y_m\})$*.*

(i) *We have the usual conventions that* Γ *is a m-context,* θ *is a m-judgement etc. as in [Se98]. This can however be as well* T*-contexts,* T*-judgements for the intermediate systems* $T = \mathrm{ML}_1^{(n)}, \mathrm{ML}_2$*. Which one, should be clear from the general context.*

Preliminaries 4.2 *Let in the following always, unless stated differently* $\vec{f} := f, g$*,* $\vec{f'} := f', g'$*, where* $f = (x, y)s$*,* $f' = (x', y')s'$*,* $g = (x, y, z)t$*,* $g' = (x', y', z')t'$ *for some variables* x, y, z, x', y', z' *and some m-terms* s, t, s', t'

Definition 4.3 *of the type theory* MLM*. We define* $\mathrm{MLM} \vdash \Gamma \Rightarrow \theta$ *in the same way as in the definition of* $\mathrm{ML_J}$ *in [Se93], (esp. with the same convention concerning the use of abstracted m-terms and -types), with the following modifications:*

- *We switch from a version which is close to [TD88] to a version close to [PSH90], (this doesn't have an influence on the proof in the following which works for most variants), i.e.:*
  - (Ref_i) *become trivial, since we treat* A type *as an abbreviation for* $A = A$, $s : A$ *as an abbreviation for* $s = s : A$.
  - (Sub_i), *can be derived and therefore omitted.*
  - *We replace all type introduction, introduction and elimination rules by their equality versions, e.g. for the case of the* Π*-type:*

$$(\Pi^{\mathrm{T}})\ \frac{A = A' \qquad x : A \Rightarrow B(x) = B'(x)}{\Pi(A, B) = \Pi(A', B')}$$

$$(\Pi^{\mathrm{I}})\ \frac{x : A \Rightarrow t(x) = t'(x) : B(x)}{\lambda(t) = \lambda(t') : \Pi(A, B)}$$

$$(\Pi^{\mathrm{E}})\ \frac{x : A \Rightarrow B(x)\ \mathrm{type} \qquad s = s' : \Pi(A, B) \qquad r = r' : A}{\mathrm{Ap}(s, r) = \mathrm{Ap}(s', r') : \Pi(A, B)}$$

- *We replace the rules for the universe by the new rules for the Mahlo universe below.*

$r \to_{\mathrm{V}} s := \widehat{\Pi}_{\mathrm{V}} x : r.s$, $r \to_{\vec{f}} s := \widehat{\Pi}_{\vec{f}} x : r.s$ *for a new variable* x.

Rules for the Mahlo-universe

Type introduction rules for V

$(\mathrm{V_I})\quad \mathrm{V} = \mathrm{V}$

$$(\mathrm{T_I})\ \frac{r = r' : \mathrm{V}}{\mathrm{T}(r) = \mathrm{T}(r')}$$

Introduction rules for V

$(\widehat{\mathrm{N}}_{k,\mathrm{V,I}})\quad \widehat{\mathrm{N}}_{k,\mathrm{V}} = \widehat{\mathrm{N}}_{k,\mathrm{V}} : \mathrm{V} \quad (k \in \omega)$

$(\widehat{\mathrm{N}}_{k,\mathrm{V},=})\quad \mathrm{T}(\widehat{\mathrm{N}}_{k,\mathrm{V}}) = \mathrm{N}_k \quad (k \in \omega)$

$$(\widehat{\Pi}_{\mathrm{V,I}})\ \frac{r = r' : \mathrm{V} \qquad x : \mathrm{T}(r) \Rightarrow s(x) = s'(x) : \mathrm{V}}{\widehat{\Pi}_{\mathrm{V}}(r, s) = \widehat{\Pi}_{\mathrm{V}}(r', s') : \mathrm{V}}$$

$$(\widehat{\Pi}_{\mathrm{V},=})\ \frac{r : \mathrm{V} \qquad x : \mathrm{T}(r) \Rightarrow s(x) : \mathrm{V}}{\mathrm{T}(\widehat{\Pi}_{\mathrm{V}}(r, s)) = \Pi x : \mathrm{T}(r).\mathrm{T}(s(x))}$$

Similarly for $\widehat{\mathrm{N}}$, $\widehat{\Sigma}$, $\widehat{\mathrm{W}}$, $\widehat{+}$, *i.*

$$(\widehat{\mathrm{U}}_{\mathrm{I}})\ \frac{\begin{array}{c} x : \mathrm{V}, y : (\mathrm{T}(x) \to \mathrm{V}) \Rightarrow f(x, y) = f'(x, y) : \mathrm{V} \\ x : \mathrm{V}, y : (\mathrm{T}(x) \to \mathrm{V}), z : \mathrm{T}(f(x, y)) \Rightarrow g(x, y, z) = g'(x, y, z) : \mathrm{V} \end{array}}{\widehat{\mathrm{U}}_{f,g} = \widehat{\mathrm{U}}_{f',g'} : \mathrm{V}}$$

$$x : \mathrm{V}, y : (\mathrm{T}(x) \to \mathrm{V}) \Rightarrow f(x,y) : \mathrm{V}$$

$$(\widehat{\mathrm{U}}_{=})\ \frac{x : \mathrm{V}, y : (\mathrm{T}(x) \to \mathrm{V}), z : \mathrm{T}(f(x,y)) \Rightarrow g(x,y,z) : \mathrm{V}}{\mathrm{T}(\widehat{\mathrm{U}}_{f,g}) = \mathrm{U}_{f,g}}$$

Rules for U

$$x : \mathrm{V}, y : (\mathrm{T}(x) \to \mathrm{V}) \Rightarrow f(x,y) = f'(x,y) : \mathrm{V}$$

$$(\mathrm{U}_{\mathrm{T}})\ \frac{x : \mathrm{V}, y : (\mathrm{T}(x) \to \mathrm{V}), z : \mathrm{T}(f(x,y)) \Rightarrow g(x,y,z) = g'(x,y,z) : \mathrm{V}}{\mathrm{U}_{f,g} = \mathrm{U}_{f',g'}}$$

$$(\mathrm{s}_{\mathrm{I}})\ \frac{\begin{array}{c} x : \mathrm{V}, y : (\mathrm{T}(x) \to \mathrm{V}) \Rightarrow f(x,y) = f'(x,y) : \mathrm{V} \\ x : \mathrm{V}, y : (\mathrm{T}(x) \to \mathrm{V}), z : \mathrm{T}(f(x,y)) \Rightarrow g(x,y,z) = g'(x,y,z) : \mathrm{V} \\ r = r' : \mathrm{U}_{f,g} \end{array}}{\mathrm{s}_{f,g}(r) = \mathrm{s}_{f',g'}(r') : \mathrm{V}}$$

Introduction-Rules for U

Let in the following $\mathrm{Ass}_{\mathrm{I}}$ *be the assumptions (for the introduction rules)*

$$\begin{array}{rcl} x : \mathrm{V}, y : (\mathrm{T}(x) \to \mathrm{V}) & \Rightarrow & f(x,y) = f'(x,y) : \mathrm{V} \ , \\ x : \mathrm{V}, y : (\mathrm{T}(x) \to \mathrm{V}), z : \mathrm{T}(f(x,y)) & \Rightarrow & g(x,y,z) = g'(x,y,z) : \mathrm{V} \ . \end{array}$$

and $\mathrm{Ass}_{=}$ *be the assumptions (for the equality rules)*

$$\begin{array}{rcl} x : \mathrm{V}, y : (\mathrm{T}(x) \to \mathrm{V}) & \Rightarrow & f(x,y) : \mathrm{V} \ , \\ x : \mathrm{V}, y : (\mathrm{T}(x) \to \mathrm{V}), z : \mathrm{T}(f(x,y)) & \Rightarrow & g(x,y,z) : \mathrm{V} \ , \end{array}$$

$$(\mathrm{Res0}_{\mathrm{I}})\ \frac{r = r' : \mathrm{U}_{s,t} \qquad x : \mathrm{T}(\mathrm{s}_{s,t}(r)) \Rightarrow s(x) = s'(x) : \mathrm{U}_{\vec{f}} \qquad \mathrm{Ass}_{\mathrm{I}}}{\mathrm{Res0}_{\vec{f}}(r,s) = \mathrm{Res0}_{\vec{f}'}(r',s') : \mathrm{U}_{\vec{f}}}$$

$$(\mathrm{Res0}_{=})\ \frac{r : \mathrm{U}_{\vec{f}} \qquad x : \mathrm{T}(\mathrm{s}_{\vec{f}}(r)) \Rightarrow s(x) : \mathrm{U}_{\vec{f}} \qquad \mathrm{Ass}_{=}}{\mathrm{s}_{\vec{f}}(\mathrm{Res0}_{\vec{f}}(r,s)) = f(\mathrm{s}_{\vec{f}}(r), \lambda x.\mathrm{s}_{\vec{f}}(s(x))) : \mathrm{V}}$$

$$(\mathrm{Res1}_{\mathrm{I}})\ \frac{\begin{array}{c} r = r' : \mathrm{U}_{\vec{f}} \qquad x : \mathrm{T}(\mathrm{s}_{\vec{f}}(r)) \Rightarrow s(x) = s'(x) : \mathrm{U}_{\vec{f}} \\ t = t' : \mathrm{T}(\mathrm{s}_{\vec{f}}(\mathrm{Res0}_{\vec{f}}(r,s))) \quad \mathrm{Ass}_{\mathrm{I}} \end{array}}{\mathrm{Res1}_{\vec{f}}(r,s,t) = \mathrm{Res1}_{\vec{f}'}(r',s',t') : \mathrm{U}_{\vec{f}}}$$

$$(\mathrm{Res1}_{=})\ \frac{r : \mathrm{U}_{\vec{f}} \qquad x : \mathrm{T}(\mathrm{s}_{\vec{f}}(r)) \Rightarrow s(x) : \mathrm{U}_{\vec{f}} \qquad t : \mathrm{T}(\mathrm{s}_{\vec{f}}(\mathrm{Res0}_{\vec{f}}(r,s))) \qquad \mathrm{Ass}_{=}}{\mathrm{s}_{\vec{f}}(\mathrm{Res1}_{\vec{f}}(r,s,t)) = g(\mathrm{s}_{\vec{f}}(r), \lambda x.\mathrm{s}_{\vec{f}}(s(x)), t) : \mathrm{V}}$$

$$(\widehat{\mathrm{N}}_{k,\vec{f},\mathrm{I}})\ \frac{\mathrm{Ass}_{\mathrm{I}}}{\widehat{\mathrm{N}}_{k,\vec{f}} = \widehat{\mathrm{N}}_{k,\vec{f}'} : \mathrm{U}_{\vec{f}}} \qquad (\widehat{\mathrm{N}}_{k,\vec{f},=})\ \frac{\mathrm{Ass}_{=}}{\mathrm{s}_{\vec{f}}(\widehat{\mathrm{N}}_{k,\vec{f}}) = \widehat{\mathrm{N}}_{k,\mathrm{V}} : \mathrm{V}}$$

$$(\widehat{\Pi}_{\vec{f},\mathrm{I}})\ \frac{r = r' : \mathrm{U}_{\vec{f}} \qquad x : \mathrm{T}(\mathrm{s}_{\vec{f}}(r)) \Rightarrow s(x) = s'(x) : \mathrm{U}_{\vec{f}} \qquad \mathrm{Ass}_{\mathrm{I}}}{\widehat{\Pi}_{\vec{f}}(r,s) = \widehat{\Pi}_{\vec{f}'}(r',s') : \mathrm{U}_{\vec{f}}}$$

$$(\widehat{\Pi}_{\vec{f},=})\ \frac{r : \mathrm{U}_{\vec{f}} \qquad x : \mathrm{T}(\mathrm{s}_{\vec{f}}(r)) \Rightarrow s(x) : \mathrm{U}_{\vec{f}} \qquad \mathrm{Ass}_{=}}{\mathrm{s}_{\vec{f}}(\widehat{\Pi}_{\vec{f}}(r,s)) = \widehat{\Pi}_{\mathrm{V}}(\mathrm{s}_{\vec{f}}(r), (x)\mathrm{s}_{\vec{f}}(s(x)) : \mathrm{V}}$$

Similarly for $\widehat{\mathrm{N}}$, $\widehat{\Sigma}$, $\widehat{\mathrm{W}}$, $\widehat{+}$, i

Definition 4.4 *(a) The intermediate system* $\mathrm{ML}_1^{(n)}$ *is defined as follows:*
The $\mathrm{ML}_1^{(n)}$*-term-constructors are the m-term-constructors except the* U*-universe-term-constructors,* V*-universe-term-constructors and Mahlo-term-constructors.*
The $\mathrm{ML}_1^{(n)}$*-type-constructors are the m-type-constructors except of* V*,* U*,* T*, but additionally with* 1*-ary constructors* Eq *and* $\mathrm{X}_1, \ldots, \mathrm{X}_n$*.*
The $\mathrm{ML}_1^{(n)}$*-terms are the m-terms, built only from* $\mathrm{ML}_1^{(n)}$*-term-constructors.*
The $\mathrm{ML}_1^{(n)}$*-types are defined as the m-types, but referring to* $\mathrm{ML}_1^{(n)}$*-types instead of m-types, only using* $\mathrm{ML}_1^{(n)}$*-type-constructors and by adding the clause: if* t *is a* $\mathrm{ML}_1^{(n)}$*-type, then* $\mathrm{Eq}(t)$ *and* $\mathrm{X}_i(t)$ *are* $\mathrm{ML}_1^{(n)}$*-type, too.*
($\mathrm{Eq}(t)$ *will be needed only to prove* $0 \neq_{\mathrm{N}} 1$*).*
The definition of $\mathrm{ML}_1^{(n)}$*-contexts, -context-pieces, -judgements are defined as usual.*
The $\mathrm{ML}_1^{(n)}$*-rules are the rules of* MLM*, but omitting all rules for the universe and adding the rules:* $\dfrac{r = s}{\mathrm{X}_i(r) = \mathrm{X}_i(s)}$, $\dfrac{r = s : \mathrm{N}_2}{\mathrm{Eq}(r) = \mathrm{Eq}(s)}$, $\mathrm{Eq}(0_2) = \mathrm{N}_0$, $\mathrm{Eq}(1_2) = \mathrm{N}_1$.
$\mathrm{ML}_1^{(n)} \vdash \Gamma \Rightarrow \theta$ *is then defined as usual.*

(b) The intermediate system ML_2 *is the restriction of* MLM *by omitting the* U*-universe-term-constructors, Mahlo-universe-constructors and the type-constructor* U*.*

Definition 4.5 *(a)* $\mathrm{V}_{\mathrm{V}} := \mathrm{V}$, $\mathrm{V}_{\vec{f}} := \mathrm{U}_{\vec{f}}$.

(b) $\mathrm{T}_{\mathrm{V}}(t) := \mathrm{T}(t)$, $\mathrm{T}_{\vec{f}}(t) := \mathrm{T}(\mathrm{s}_{\vec{f}}(t))$.

(c) $\mathcal{P}_{\mathrm{V}}(\mathrm{N}) := \mathrm{N} \to \mathrm{V}_v$.

Definition 4.6 *Let* $v = \vec{f}$ *or* $v = \mathrm{V}$.

(a) For $\mathrm{ML}_1^{(n)}$*-types* A*, m-types* $A_1, \ldots, A_n$ *and variables* $x_1, \ldots, x_n$ *we define, with* $B_i := (x_i)A_i$ *the m-type* $A[\mathrm{X}_1 := B_1, \ldots, \mathrm{X}_n := B_n]$*, short* $A[\vec{\mathrm{X}} := \vec{B}]$ *inductively by:*
Is C *an* $\mathrm{ML}_1^{(n)}$*-type-constructor,* $C \notin \{\mathrm{Eq}, \mathrm{X}_1, \ldots, \mathrm{X}_n\}$*, then* $C(t_1, \ldots, t_n)[\vec{\mathrm{X}} := \vec{B}] := C(t_1[\vec{\mathrm{X}} := \vec{B}], \ldots, t_n[\vec{\mathrm{X}} := \vec{B}])$,
where $((x)D)[\vec{\mathrm{X}} := \vec{B}]) := (x)(D[\vec{\mathrm{X}} := \vec{B}])$*, and for* $\mathrm{ML}_1^{(n)}$*-terms* t*,* $t[\vec{\mathrm{X}} := \vec{B}] := t$.
$\mathrm{X}_i(t)[\vec{\mathrm{X}} := \vec{B}] := A_i[x_i := t]$.
$\mathrm{Eq}(t)[\vec{\mathrm{X}} := \vec{B}] := \mathrm{T}(\mathrm{C}_2(t, \widehat{\mathrm{N}}_{0,\mathrm{V}}, \widehat{\mathrm{N}}_{1,\mathrm{V}}))$.
$(\Gamma \Rightarrow \theta)[\vec{\mathrm{X}} := \vec{B}]$ *is the result of applying this operation to the (outermost) types in* $\Gamma \Rightarrow \theta$.

(b) For $\mathrm{ML}_1^{(n)}$*-type-constructors* C *we define* $\gamma_v(C)$*:* $\gamma_v(\mathrm{N}_k) := \widehat{\mathrm{N}}_{k,v}$, $\gamma_v(\mathrm{N}) := \widehat{\mathrm{N}}_v$, $\gamma_v(\Sigma) := \widehat{\Sigma}_v$, $\gamma_v(\Pi) := \widehat{\Pi}_v$, $\gamma_v(\mathrm{W}) := \widehat{\mathrm{W}}_v$, $\gamma_v(\mathrm{I}) := \widehat{\mathrm{I}}_v$, $\gamma_v(+) := \widehat{+}_v$, $\gamma_v(C) := C$ *otherwise.*

(c) For $\mathrm{ML}_1^{(n)}$*-types* A*, and variables* $Y_1, \ldots, Y_n$ *(which will later range over elements of* $\mathrm{N} \to \mathrm{V}_v$*) we define* $\gamma_v(A)[\mathrm{X}_1 := Y_1, \ldots, \mathrm{X}_n := Y_n]$*, where we abbreviate* $[\mathrm{X}_1 := Y_1, \ldots, \mathrm{X}_n := Y_n]$ *by* $[\vec{\mathrm{X}} := \vec{Y}]$*:*
If $C \notin \{\mathrm{Eq}, \mathrm{X}_1, \ldots, \mathrm{X}_n\}$*, then*
$\gamma_v(C(t_1, \ldots, t_n)[\vec{\mathrm{X}} := \vec{Y}] := \gamma_v(C)(\gamma_v(t_1)[\vec{\mathrm{X}} := \vec{Y}], \ldots, \gamma_v(t_n)[\vec{\mathrm{X}} := \vec{Y}])$,
where $\gamma_v((x)D)[\vec{\mathrm{X}} := \vec{Y}]) := (x)(\gamma_v(D)[\vec{\mathrm{X}} := \vec{Y}])$,
and for $\mathrm{ML}_1^{(n)}$*-terms* t*,* $\gamma_v(t)[\vec{\mathrm{X}} := \vec{Y}] := t$.
$\gamma_v(\mathrm{X}_i(t))[\vec{\mathrm{X}} := \vec{Y}] := Y_i t$,
$\gamma_v(\mathrm{Eq}(t))[\vec{\mathrm{X}} := \vec{Y}] := \mathrm{C}_2(t, \widehat{\mathrm{N}}_{0,v}, \widehat{\mathrm{N}}_{1,v})$.

(d) We define $\gamma_v(\theta)[\vec{\mathrm{X}} := \vec{Y}]$ *for* $\mathrm{ML}_1^{(n)}$*-judgements* θ*:*
$\gamma_v(\text{context})[\vec{\mathrm{X}} := \vec{Y}] := \{\text{context}\}$,

$\gamma_v(s = t : A) := \{\gamma_v(A)[\vec{\mathrm{X}} := \vec{Y}] : \mathrm{V}_v, s = t : \mathrm{T}_v(\gamma_v(A))\}$,
$\gamma_v(A = B) := \{\gamma_v(A)[\vec{\mathrm{X}} := \vec{Y}] = \gamma_v(B)[\vec{\mathrm{X}} := \vec{Y}] : \mathrm{V}_v\}$.
$\gamma_v(x_1 : A_1, \ldots, x_n : A_n) := x_1 : \mathrm{T}(\gamma_v(A_1)), \ldots, x_n : \mathrm{T}(\gamma_v(A_n))$.
$\gamma_v(x_1 : A_1, \ldots, x_n : A_n \Rightarrow \theta)[\vec{\mathrm{X}} := \vec{Y}] :=$
$\{x_1 : \mathrm{T}_v(\gamma_v(A_1)), \ldots, x_{i-1} : \mathrm{T}_v(\gamma_v(A_{i-1})) \Rightarrow \gamma_v(A_i) : \mathrm{V}_v \mid i = 1, \ldots, n\} \cup \{x_1 : \mathrm{T}_v(\gamma_v(A_1)), \ldots, x_n : \mathrm{T}_v(\gamma_v(A_n)) \Rightarrow \theta' | \theta' \in \gamma_v(\theta)\}$.

(e) *For* ML_2*-term-constructors* C *we define* $\delta_{\vec{f}}(C)$: $\delta_{\vec{f}}(\widehat{\mathrm{N}}_{k,\mathrm{V}}) := \widehat{\mathrm{N}}_{k,\vec{f}}$, $\delta_{\vec{f}}(\widehat{\mathrm{N}}) := \widehat{\mathrm{N}}_{\vec{f}}$, $\delta_{\vec{f}}(\widehat{\Sigma}_{\mathrm{V}}) := \widehat{\Sigma}_{\vec{f}}$, $\delta_{\vec{f}}(\widehat{\Pi}_{\mathrm{V}}) := \widehat{\Pi}_{\vec{f}}$, $\delta_{\vec{f}}(\widehat{\mathrm{W}}_{\mathrm{V}}) := \widehat{\mathrm{W}}_{\vec{f}}$, $\delta_{\vec{f}}(\widehat{\mathrm{I}}_{\mathrm{V}}) := \widehat{\mathrm{I}}_{\vec{f}}$, $\delta_{\vec{f}}(\widehat{+}_{\mathrm{V}}) := \widehat{+}_{\vec{f}}$, $\delta_{\vec{f}}(C) := C$ *otherwise.*
For ML_2*-type-constructors* C *we define* $\delta_{\vec{f}}(C)$: $\delta_{\vec{f}}(\mathrm{N}_k) := \widehat{\mathrm{N}}_{k,\mathrm{V}}$, $\delta_{\vec{f}}(\mathrm{N}) := \widehat{\mathrm{N}}_{\mathrm{V}}$, $\delta_{\vec{f}}(\Sigma) := \widehat{\Sigma}_{\mathrm{V}}$, $\delta_{\vec{f}}(\Pi) := \widehat{\Pi}_{\mathrm{V}}$, $\delta_{\vec{f}}(\mathrm{W}) := \widehat{\mathrm{W}}_{\mathrm{V}}$, $\delta_{\vec{f}}(\mathrm{I}) := \widehat{\mathrm{I}}_{\mathrm{V}}$, $\delta_{\vec{f}}(+) := \widehat{+}_{\mathrm{V}}$, $\delta_{\vec{f}}(\mathrm{V}) := \mathrm{U}_{\vec{f}}$, $\delta_{\vec{f}}(\mathrm{T}) := \mathrm{s}_{\vec{f}}$, $\delta_{\vec{f}}(C) := C$ *otherwise.*

(f) $\rho_{\vec{f}}(C) := \delta_{\vec{f}}(C)$ *for term-constructors* C, $\rho_{\vec{f}}(\mathrm{V}) := \mathrm{T}(\mathrm{U}_{\vec{f}})$, $\rho_{\vec{f}}(\mathrm{T}) := \mathrm{T}_{\vec{f}}$, $\rho_{\vec{f}}(C) := C$ *otherwise.*

(g) *For* ML_2*-terms or types* t, $\delta_{\vec{f}}(t)$, $\rho_{\vec{f}}(t)$ *is the result of replacing all constructors* C *in* t *by* $\delta_{\vec{f}}(C)$ *or* $\rho_{\vec{f}}(C)$ *respectively.*

(h) *We define* $\delta_{\vec{f}}(\theta)$ *for* $\mathrm{ML}_1^{(n)}$*-judgements* θ: $\delta_{\vec{f}}(\mathrm{context}) := \{\mathrm{context}\}$, $\delta_{\vec{f}}(s = t : A) := \{\delta_{\vec{f}}(A) : \mathrm{U}_{\vec{f}}, \delta_{\vec{f}}(s) = \delta_{\vec{f}}(t) : \mathrm{T}_{\vec{f}}(\delta_{\vec{f}}(A))\}$, $\delta_{\vec{f}}(A = B) := \{\delta_{\vec{f}}(A) = \delta_{\vec{f}}(B) : \mathrm{U}_{\vec{f}}\}$.
$\delta_{\vec{f}}(x_1 : A_1, \ldots, x_n : A_n \Rightarrow \theta) := \{x_1 : \mathrm{T}_{\vec{f}}(\delta_{\vec{f}}(A_1)), \ldots, x_{i-1} : \mathrm{T}_{\vec{f}}(\delta_{\vec{f}}(A_{i-1})) \Rightarrow \delta_{\vec{f}}(A_i) : \mathrm{U}_{\vec{f}} \mid i = 1, \ldots, n\} \cup \{x_1 : \mathrm{T}_{\vec{f}}(\delta_{\vec{f}}(A_1)), \ldots, x_n : \mathrm{T}_{\vec{f}}(\delta_{\vec{f}}(A_n)) \Rightarrow \theta' | \theta' \in \delta_{\vec{f}}(\theta)\}$.

(i) $\rho_{\vec{f}}(\Gamma \Rightarrow \theta)$ *is the result of replacing all terms and types* t *in* $\Gamma \Rightarrow \theta$ *by* $\rho_{\vec{f}}(t)$.

(j) *If* $v = \mathrm{V}$ $\mathrm{Assumptions}_\Delta(v)$ *is empty, and if* $v = \vec{f}$ *as usual,*
$\mathrm{Assumptions}_\Delta(v)$ *is the condition* $\mathrm{MLM} \vdash \Delta, x_1 : \mathrm{V}, y : \mathrm{T}(x_1) \to \mathrm{V} \Rightarrow f(x_1, y) : \mathrm{V}$ *and* $\mathrm{MLM} \vdash \Delta, x_1 : \mathrm{V}, y : \mathrm{T}(x_1) \to \mathrm{V}, z : \mathrm{T}(f(x_1, y)) \Rightarrow g(x_1, y, z) : \mathrm{V}$.

Lemma 4.7 *Assume* $v = \mathrm{V}$, $\mathrm{Assumptions}_\Delta(v)$.

(a) *If* $\mathrm{ML}_1^{(n)} \vdash \Gamma \Rightarrow \theta$, $\mathrm{MLM} \vdash \Delta, x_i : \mathrm{N} \Rightarrow B_i$ type *($i = 1, \ldots, n$), then* $\mathrm{MLM} \vdash \Delta, (\Gamma \Rightarrow \theta)[\mathrm{X}_1 := (x_1)B_1, \ldots, \mathrm{X}_n := (x_n)B_n]$. *The same holds with* MLM *replaced by* $\mathrm{ML}_1^{(n)}$ *or* ML_2*; only in the case* $T = \mathrm{ML}_1^{(n)}$*, we have to modify the definition of* $t[\vec{\mathrm{X}} := \vec{B}]$ *by* $\mathrm{Eq}(t)[\vec{\mathrm{X}} := \vec{B}] := \mathrm{Eq}(t)$.

(b) *If* $\mathrm{ML}_1^{(n)} \vdash \Gamma \Rightarrow \theta$, $(\Gamma_1 \Rightarrow \theta_1) \in \gamma_v(\Gamma \Rightarrow \theta)[\vec{\mathrm{X}} := \vec{Y}]$*, then* $\mathrm{MLM} \vdash \Delta, Y_1 : \mathcal{P}_v(\mathrm{N}), \ldots, Y_n : \mathcal{P}_v(\mathrm{N}), \Gamma \Rightarrow \theta_1$. *The same holds in the case* $v = \mathrm{V}$ *with* MLM *replaced by* ML_2.

(c) *If* $\mathrm{ML}_1^{(n)} \vdash \Gamma \Rightarrow A :$ type*, then* $\mathrm{MLM} \vdash \Delta, Y_1 : \mathcal{P}_v(\mathrm{N}), \ldots, Y_n : \mathcal{P}_v(\mathrm{N}), \Gamma \Rightarrow r : \mathrm{T}_v(\gamma_v(A))[\vec{\mathrm{X}} := \vec{Y}] \leftrightarrow A[\vec{\mathrm{X}} := \vec{Y}^{\mathrm{V,Cl}}]$ *for some* r*, where* $\vec{Y}^{\mathrm{V,Cl}} := Y_1^{\mathrm{V,Cl}}, \ldots, Y_n^{\mathrm{V,Cl}}$.

(d) *If* $\mathrm{ML}_2 \vdash \Gamma \Rightarrow \theta$*, then* $\mathrm{MLM} \vdash \Gamma \Rightarrow \theta$.

(e) *If* $v = \vec{f}$ *as in the assumptions of this lemma,* $(\Gamma_1 \Rightarrow \theta_1) \in \delta_{\vec{f}}(\Gamma \Rightarrow \theta)$*, then* $\mathrm{MLM} \vdash \Delta, \Gamma_1 \Rightarrow \theta_1$ *and* $\mathrm{MLM} \vdash \Delta, \rho_{\vec{f}}(\Gamma \Rightarrow \theta)$.

Proof: By an easy induction on the $\mathrm{ML}_1^{(n)}$, ML_2-derivations. In (c) we conclude for instance in the case of Π-type-introduction, having $h_1 : \mathrm{T}_v(\gamma_v(A)) \to A$, $h_2 : B \to \mathrm{T}_v(\gamma_v(B))$ that $\lambda h_3.\lambda y.h_2[x := h_1 y](h_3(h_1 y)) : ((\Pi x : A.B) \to \mathrm{T}_v(\gamma_v(\Pi x : A.B)))$. In the case of the W-type, we use induction over the W-type.

We use all the abbreviations from Sect. 3.2 and the definitions and lemmata in Sect. 3.3 - 3.5 of [Se98]. The changes are

- We replace the underlined constructors by their non-underlined version.
- All proofs and definitions can be carried out in $\mathrm{ML}_1^{(n)}$.
- We add the following definition and lemma:

Definition 4.8 *Assume* $v = \mathrm{V}$ *or* $v = \vec{f}$.

(a) $t \in_{\mathrm{V}} A := At$, $t \in_{\vec{f}} A := At$. *(Note that, if* $A : \mathcal{P}_v(\mathrm{N})$, $t : \mathrm{N}$, *then* $t \in_v A : \mathrm{V}_v$*).*

(b) $t^{\mathrm{V,Cl}} := \{y : \mathrm{N} \mid \mathrm{T}(ty)\}$, $t^{\vec{f},\mathrm{V}} := \lambda y.\mathrm{s}_{\vec{f}}(ty)$, $t^{\vec{f},\mathrm{Cl}} := (t^{\vec{f},\mathrm{V}})^{\mathrm{V,Cl}}$.

Lemma 4.9 *Assume* $v = \mathrm{V}$ *or* $v = \vec{f}$, $\mathrm{Assumptions}_\Delta(v)$, $\mathrm{ML}_1^{(n)} \vdash \Delta \Rightarrow A \in \mathrm{Cl}(\mathrm{N})$.

(a) $\mathrm{MLM} \vdash \Delta, Y_1 : \mathcal{P}_v(\mathrm{N}), \ldots, Y_n : \mathcal{P}_v(\mathrm{N}) \Rightarrow \gamma_v(A)[\vec{\mathrm{X}} := \vec{Y}] : \mathcal{P}_v(\mathrm{N})$.

(b) $\mathrm{MLM} \vdash \Delta, Y_1 : \mathcal{P}_v(\mathrm{N}), \ldots, Y_n : \mathcal{P}_v(\mathrm{N}) \Rightarrow \gamma_v(A)^{v,\mathrm{Cl}}[\vec{\mathrm{X}} := \vec{Y}] \cong A[\vec{\mathrm{X}} := \vec{Y}^{\mathrm{V,Cl}}]$.

Proof: easy.

5 The Well-ordering Proofs

The definition of $\mathrm{M}(A)$, $\tau^A(a)$, $\mathcal{A}^A(B)$, $W(A)$, $\mathrm{Ag}(A)$, A^+, $\mathrm{Prog}(A, B)$, $\mathrm{Prog}(A)$, weakly upward and downward closed can be done as in Sect. 4.2 - 4.4 of [Se98]. Further nearly all the lemmata in these section go through again. We have the following changes:

- We can do all the proofs in $\mathrm{ML}_1^{(n)}$. We only replace the assumptions of $A, B, C, \ldots \in \mathrm{Cl}(\mathrm{N})$ by: let $A = (x)X_{i_1}(x)$, $B = (x)X_{i_2}(x) \ldots$.
- We have to replace I by M.
- Lemma 4.17 in [Se98] is replaced by Lemma 5.2 below.
- We add Lemma 5.1 below, which is proved in $\mathrm{ML}_1^{(n)}$.

Lemma 5.1 *Let* $A = X_i$ *for some* $i \le n$.

(a) $(\tau^A(\psi_\kappa b) \subset A \wedge \psi_\kappa b \preceq c \preceq \kappa) \to \mathrm{C}^{\psi_\kappa b}(A) \cap \mathrm{C}_\kappa(b) \subset \mathrm{C}^c(A)$.

(b) *If* $a \in \mathrm{C}^b(A)$, $A \subset \mathrm{M}(A)$, $A \cap \tilde{b}$ *weakly downward closed,* $b \prec \mathrm{M}$, *then* $\mathrm{sc}_{\mathrm{M}}(a) \subset \mathrm{C}^b(A)$.

(c) $a \in \mathrm{C}_\kappa(c) \Rightarrow \mathrm{sc}_{\mathrm{M}}(a) \subset \mathrm{C}_\kappa(c)$.

(d) $\psi_{\psi_{\mathrm{M}} a} b \in \mathsf{OT} \Rightarrow \mathrm{sc}_{\mathrm{M}}(a) \prec \psi_{\psi_{\mathrm{M}} a} b$.

(e) *If* $a, \psi_\kappa b \in \mathsf{OT}$, $\mathrm{sc}_{\mathrm{M}}(a) \prec \psi_\kappa b$ *and* $\mathrm{G}_{\mathrm{M}}(a) \prec b$, *then* $a \in \mathrm{C}_\kappa(b)$.

Proof: (a): We show $\forall x \in \mathrm{C}^{\psi_\kappa b}(A) \cap \mathrm{C}_\kappa(b).x \in \mathrm{C}^c(A)$ by induction on length(x). If $x \prec \psi_\kappa b$, it follows $x \in \mathrm{C}^{\psi_\kappa b}(A) \cap \psi_\kappa b = \tau^A(\psi_\kappa b) \subset A \cap c \subset \mathrm{C}^c(A)$. If $\psi_\kappa b \preceq x$, it follows $\kappa \preceq x$, and, if in this case $x = 0, \mathrm{M}$ or $x =_{\mathrm{NF}} x_1 + x_2, \varphi_{x_1} x_2, \Omega_{x_1}$, follows the assertion by IH or directly, and, if $x = \psi_\pi d$, $\pi, d \in \mathrm{C}_\pi(d)$, it follows $c \preceq \kappa \prec \pi$, $\pi, d \in \mathrm{C}^{\psi_\kappa b}(A) \cap \mathrm{C}_\kappa(b) \subset \mathrm{C}^c(A)$, $x \in \mathrm{C}^c(A)$.
(b), (c): easy.
(d): $\psi_{\mathrm{M}} a \in \mathrm{C}_{\psi_{\mathrm{M}} a}(b)$, $a \in \mathrm{C}_{\psi_{\mathrm{M}} a}(b)$, $\mathrm{sc}_{\mathrm{M}}(a) \subset \mathrm{C}_{\psi_{\mathrm{M}} a}(b)$, further $\mathrm{sc}_{\mathrm{M}}(a) \prec \psi_{\mathrm{M}} a$, therefore $\mathrm{sc}_{\mathrm{M}}(a) \subset \mathrm{C}_{\psi_{\mathrm{M}} a}(b) \cap \psi_{\mathrm{M}} a \cong \psi_{\psi_{\mathrm{M}} a} b$
(e): Induction on length(a). If $a = 0, \mathrm{M}$, or $a =_{\mathrm{NF}} a_1 + a_2, \varphi_{a_1} a_2, \Omega_{a_1}$, the assertion follows by IH or directly. If $a = \psi_\pi c$, $\pi \preceq \mathrm{M}$, it follows $a \prec \psi_\kappa b$, $a \in \mathrm{C}_\kappa(b)$, and if $a = \psi_\pi c$, $\mathrm{M} \prec \pi$, follows the assumption by IH for π, c.

Lemma 5.2 *(replaces Lemma 4.17 in [Se98])*

(a) $0 \in W(A)$.

(b) If $A \cap \kappa \cong W(A) \cap \kappa$, $A \subset \mathrm{M}(A)$, *then* $W(A) \cap \kappa^+$ *is downward closed, upward closed bounded by* κ^+.

(c) Assume $A \subset \mathrm{M}(A)$. *Then* $\forall \kappa, y \in W(A). \forall z \in \mathsf{OT}.(z = \max\{\kappa, y\} \wedge W(A) \cap \tilde{z} \cong A \cap \tilde{z} \wedge \kappa \in \mathsf{R} \wedge \kappa, y \in \mathrm{C}_\kappa(y)) \to \psi_\kappa y \in W(A)$.

Proof: (a): trivial. (b): as in the proof of Lemma 4.17 in [Se98].
(c): Assume $A \subset \mathrm{M}(A)$, $\tau \in \mathsf{Car} \cap W(A)$, $W(A) \cap \tau \cong A \cap \tau$. We show by Ind$(y \in W(A))$

$$\forall y \in W(A). \forall \kappa \in \mathsf{R}.(y \prec \tau^+ \wedge y, \kappa \in \mathrm{C}_\kappa(y) \wedge \psi_\kappa y \prec \tau^+$$
$$\wedge(\kappa = \mathrm{M} \vee \exists z \in \mathsf{OT}.(\kappa = \psi_{\mathrm{M}} z \wedge z \in \mathrm{C}^y(A) \cap \mathrm{C}_{\mathrm{M}}(z)) \vee \kappa^- \in W(A)|\tau)$$
$$\to \psi_\kappa y \in W(A).$$

Then follows with $\tau := \tilde{z}$ the assertion.
Assume y according to Induction, κ, y as in the premise of the assertion.
We note first: if $c \in W(A) \cap y$, then $c \in \tau^A(y)$ (if $c \prec \tau$, then $c \in A \cap y$, otherwise $\tilde{c} = \tilde{y}$, $c \in \mathrm{C}^c(A) \cap y \cong \tau^A(y))$ and $c \in \mathrm{C}^c(A) \cap y$.
We show

$$(*) \qquad \forall x' \in \mathrm{C}_\kappa(y) \cap \mathrm{C}^{\psi_\kappa y}(A) \cap \tau^+ . x' \in W(A)$$

by side-induction on length(x') and assume x' according to induction, $x' \in \mathrm{C}_\kappa(y) \cap \mathrm{C}^{\psi_\kappa y}(A) \cap \tau^+$.
If $x' = 0, \mathrm{M}$, follows the assertion by (b). If $x' =_{\mathrm{NF}} x_1 + x_2, \varphi_{x_1} x_2$ or Ω_{x_1}, it follows by IH $x_i \in W(A)$, by (b) $x' \in W(A)$.
Case $x' = \psi_\pi x_1$.
Subcase $x' \prec \psi_\kappa y$. Then $x' \in A \vee (\psi_\kappa y \prec \pi \in \mathsf{I} \wedge \pi, x_1 \in \mathrm{C}^{\psi_\kappa y}(A))$.
If $x' \in A$, it follows $x' \in A \cap \tau \subset W(A)$.
Assume now $\pi, x_1 \in \mathrm{C}^{\psi_\kappa y}(A)$, $\psi_\kappa y \prec \pi \in \mathsf{I}$.
If $\kappa = \Omega_{c+1}$, it follows $x' \in \mathrm{C}^{\kappa^-}(A) \cap \kappa^- \subset W(A)$.
Otherwise $\kappa, \pi \in \mathsf{I} \wedge x' \prec \psi_\kappa y \prec \pi$. Therefore $\psi_\pi x_1 \preceq \mathrm{sc}_\kappa(y) \vee (\pi = \kappa \wedge x_1 \prec y \wedge \mathrm{sc}_\pi(x_1) \prec \psi_\kappa y) \vee (\psi_\pi x_1 \prec \kappa \prec \pi \wedge \mathrm{sc}_\pi(x_1) \prec \psi_\kappa y)$.
Subsubcase $\psi_\pi x_1 \preceq \mathrm{sc}_\kappa(y)$. $y \in W(A)$, $\mathrm{sc}_\kappa(y) \subset \tau^A(y) \cup \{y\} \subset W(A)$, $\mathrm{sc}_\kappa(y) \subset \psi_\kappa y$. Let $a := \max(\mathrm{sc}_\kappa(y))$. Then $\psi_\pi x_1 \in \mathrm{C}^{\psi_\kappa y}(A)|a \subset \mathrm{C}^a(A)|a \cong \tau^A(a) \cup \{a\} \subset W(A)$.
Subsubcase $\pi = \kappa \wedge x_1 \prec y \wedge \mathrm{sc}_\pi(x_1) \prec \psi_\kappa y$: $\mathrm{G}_\kappa(x_1) \prec x_1 \prec y \prec \tau^+$, therefore $x_1 \in \mathrm{C}_\kappa(y) \cap \tau^+$, $x_1 \in \mathrm{C}^{\psi_\kappa y}(A)$. By side-IH $x_1 \in W(A) \cap \tau^+$. Therefore $x_1 \in \tau^A(y)$, and by main-IH $x' \in W(A)$.
Subsubcase $\psi_\pi x_1 \prec \kappa \prec \pi \wedge \mathrm{sc}_\pi(x_1) \prec \psi_\kappa y$: Then $\kappa = \psi_{\mathrm{M}} b$ for some b, $b \in \mathrm{C}^y(A) \vee \kappa \in W(A)$. By $\psi_\pi x_1 \prec \psi_{\mathrm{M}} b \prec \pi$ it follows $\psi_\pi x_1 \preceq \mathrm{sc}_{\mathrm{M}}(b) \vee (\pi = \mathrm{M} \wedge x_1 \prec b \wedge \mathrm{sc}_\pi(x_1) \prec \psi_{\mathrm{M}} b)$.
Subsubsubcase $\psi_\pi x_1 \preceq \mathrm{sc}_{\mathrm{M}}(b)$: Let $a := \max(\mathrm{sc}_{\mathrm{M}}(b))$. If $\kappa \in W(A)$, then $\kappa \in \mathrm{C}^\kappa(A)$,

$b \in \mathrm{C}^\kappa(A)$, $a \in \mathrm{C}^\kappa(A)$, $b \in \mathrm{C_M}(b)$, $a \in \mathrm{C_M}(b) \cap \mathrm{M} \cong \psi_{\mathrm{M}} b$, $\kappa \in \mathrm{C}_\kappa(y), a \in \mathrm{C}_\kappa(y) \cap \kappa \cong \psi_\kappa y$ $a \in \tau^A(\kappa) \subset W(A)$, $a \in W(A) \cap \psi_\kappa y$, $\psi_\pi x_1 \in \mathrm{C}^{\psi_\kappa y}(A) \cap a \subset \tau^A(a) \subset W(A)$. If $b \in \mathrm{C}^y(A)$, it follows $a \in \mathrm{C}^y(A)$, by $\kappa \in \mathrm{C}_\kappa(y)$ $a \preceq b \prec y$, again $a \in \tau^A(y) \subset W(A)$, as before $\psi_\pi x_1 \in W(A)$.

Subsubsubcase ($\pi = \mathrm{M} \wedge x_1 \prec b \wedge \mathrm{sc}_\pi(x_1) \prec \psi_{\mathrm{M}} b$): $\mathrm{sc_M}(x_1) \prec \psi_\kappa y$, $\mathrm{G_M}(x_1) \prec x_1 \prec b \prec y \prec \tau^+$, by Lemma 5.1 (e) it follows $x_1 \in \mathrm{C}_\kappa(y) \cap \mathrm{C}^{\psi_\kappa y}(A) \cap \tau^+$, by side-IH $x_1 \in W(A)$, $x_1 \prec y$, therefore $x_1 \in \tau^A(y)$, by IH $x' \in W(A)$.

Subcase $\widetilde{\psi_\kappa y} = x'$. $x' \neq \psi_\kappa y$, therefore $\kappa \notin \mathsf{I}$, $x' = \kappa^- \in W(A)$.

Subcase $\widetilde{\psi_\kappa y} \prec x' \prec \psi_\kappa y$: Then $\kappa = \Omega_{d+1} = \pi$ for some d, $x_1 \prec y$, $x_1 \in \mathrm{C}^{\psi_\kappa y}(A) \cap \mathrm{C}_\kappa(y) \cap \tau^+$, by side-IH $x_1 \in W(A)$, $x_1 \in \tau^A(y)$, by main-IH follows the assertion.

Subcase $\psi_\kappa y \preceq x'$. Then $\kappa \preceq x'$. $\pi, x_1 \in \mathrm{C}^{\psi_\kappa y}(A) \cap \mathrm{C}_\kappa(y)$, $x_1 \prec y \prec \tau^+$, by side-IH $x_1 \in W(A)$, $x_1 \in \tau^A(y)$.

Subsubcase $\tau^+ \prec \pi = \mathrm{M}$: by main-IH follows the assertion.

Subsubcase $\tau^+ \prec \pi = \psi_{\mathrm{M}} b$: $b \prec x_1 \prec y \prec \tau^+$, $b \in \mathrm{C}^{\psi_\kappa y}(A) \cap \mathrm{C}_\kappa(y) \cap \tau^+$, by side IH $b \in W(A)$, $b \in \mathrm{C}^y(A) \cap \mathrm{C_M}(b)$, by main-IH $x' \in W(A)$.

Subsubcase otherwise: Then $\pi \preceq \tau^+$. $\pi = \tau^+$, $\pi^- \in W(A)$, or $\pi \prec \tau^+$, by side-IH $\pi \in W(A)$, again $\pi^- \in W(A)$. By main-IH it follows $x' \in W(A)$.

The proof of (*) is now complete.

We have therefore $\mathrm{C}^{\psi_\kappa y}(A) \cap \psi_\kappa y \subset W(A)$. Further, if $y \prec \widetilde{\psi_\kappa y}$, $y \in W(A) \cap \tau \cap \widetilde{\psi_\kappa y} \subset \mathrm{C}^{\psi_\kappa y}(A)$, otherwise $y \in \mathrm{M}(A)$, $y \in \mathrm{C}^y(A) \subset \mathrm{C}^{\psi_\kappa y}(A)$.

If $\kappa = \mathrm{M}$, $\kappa \in \mathrm{C}^{\psi_\kappa y}(A)$.

If $\kappa = \psi_{\mathrm{M}} b$, $b \in \mathrm{C}^y(A)$,it follows $b \prec y \prec \tau^+$, $b \in W(A)$. If $\mathrm{M} \preceq \tau^+$, it follows $\psi_{\mathrm{M}} b \in W(A)$ by main IH. Otherwise, if $b \prec \tau$, it follows $b \in A \subset \mathrm{C}^{\psi_\kappa y}(A)$ and, if $\tau \preceq b$, we have $b \in \mathrm{C}^b(A) \subset \mathrm{C}^{\psi_\kappa y}(A)$, in both cases therefore $\psi_{\mathrm{M}} b \in \mathrm{C}^{\psi_\kappa y}(A)$.

If $\kappa = \kappa^- \in W(A)|\tau$, it follows $\kappa \in \mathrm{C}^\kappa(A) \subset \mathrm{C}^{\psi_\kappa y}(A)$.

If $\kappa \neq \kappa^- \in W(A)|\tau$, it follows $\kappa = \Omega_{d+1}$ for some d, $\kappa^- = \Omega_d \in \mathrm{C}^{\kappa^-}(A) \cong \mathrm{C}^{\psi_\kappa y}(A)$, $\kappa \in \mathrm{C}^{\psi_\kappa y}(A)$.

In any case therefore $\kappa, y \in \mathrm{C}^{\psi_\kappa y}(A)$, $\psi_\kappa y \in \mathrm{C}^{\psi_\kappa y}(A)$, $\psi_\kappa y \in \mathrm{M}(A)$, $\psi_\kappa y \in \mathcal{A}^A(W(A)) \subset W(A)$.

We define $\mathcal{W}$ and prove Lemma 4.26 - 4.29 as in [Se98]. The modifications with respect to [Se98] are that we can carry out all the proofs in ML_2, and can replace the assumptions $A, B, \ldots \in \mathrm{Cl(N)}$ by $A, B, \ldots : \mathcal{P}(\mathrm{N})$ (replacing $s \in A$ by $\mathrm{T}(As)$ etc.) The following Lemma 5.3, 5.4, 5.5, 5.6 and 5.7 can be proved in ML_2. $\mathcal{W}_{\vec{f}} := \delta_{\vec{f}}(\mathcal{W})$.

Lemma 5.3 *Assume* $a \in (\mathrm{M}(\mathcal{W}) \setminus \mathsf{I}) \cap \mathrm{M}$, $B : \mathcal{P}(\mathrm{N})$, $B^{\mathrm{V,Cl}} \subset \mathcal{W}|a$ *such that* $\tau^{\mathcal{W}}(a) \preceq B^{\mathrm{V,Cl}}$. *Then* $a \in \mathcal{W}$.

Proof: Let $B' := B^{\mathrm{V,Cl}}$, $\hat{B} := \Sigma x : \mathrm{N}.x \in B'$. From the assertion we conclude that there is some $g : \hat{B} \to \mathcal{P}(\mathrm{N})$ such that $\forall y : \hat{B}.\mathrm{Ag}((g\ y)^{\mathrm{V,Cl}}) \wedge y0 \in (g\ y)^{\mathrm{V,Cl}}$. Let $\hat{g}(y) := (g\ y)^{\mathrm{V,Cl}}$, $C := \{x : \mathrm{N} \mid \exists y \in \hat{B}.x \in \hat{g}(y)\} \cap a\}$. By [Se98], Lemma 4.28 it follows $\mathrm{Ag}(C)$.

We have $\tau^{\mathcal{W}}(a) \subset \mathcal{W}$: If $y \in \tau^{\mathcal{W}}(a)$, then $y \in \mathrm{C}^a(\mathcal{W}) \cap a$, $y \preceq x \in \mathcal{W}$ for some $x \in B' \subset \mathcal{W}$, $x \preceq a$, $y \in \mathrm{C}^a(\mathcal{W})|x \subset \tau^{\mathcal{W}}(x) \cup \{x\} \subset \mathcal{W}$.

We show $C \cong \mathcal{W} \cap a$: "$\subset$" is obvious. "$\supset$": Assume $y \in \mathcal{W} \cap a$. Then $y \in \tau^{\mathcal{W}}(a)$, $y \preceq z$ for some $z \in B'$, $z \in \hat{g}(\mathrm{p}(z,p)) \sqsubseteq \mathcal{W}$ for some $p : z \in B'$, $y \in \mathcal{W}$, therefore $y \in g(\mathrm{p}(z,p)) \cap a \subset C$.

We show $\forall d \in W(C).d \prec a \to d \in C$, (i.e. $W(C) \cap a \subset C$) by $\mathrm{Ind}(d \in W(C))$.

Assume d according to induction, $d \prec a$. Then $\tau^C(d) \subset C$, $d \in \mathrm{C}^d(C)$. We show $\exists e \in \mathcal{W}.d \preceq e$:

If $\tilde{d} \prec d$, it follows $\tilde{d} \in \mathrm{C}^d(C) \cap d \subset C \subset \mathcal{W}$, $d \prec d^+ \in \mathcal{W}$.

If $\tilde{d} = d =_{\mathrm{NF}} \Omega_{d'}$, it follows $d \in \mathrm{C}^d(C)$, $d' \in \mathrm{C}^d(C) \cap d \subset C \cap a \subset \mathrm{C}^a(C)$, $d = \tilde{d} \in \mathrm{C}^a(C) \cap a \cong \mathrm{C}^a(\mathcal{W}) \cap a \subset \mathcal{W}$.

If $\tilde{d} = d = \psi_{\mathrm{M}} e_1$, it follows by Lemma 5.1 (a) $e_1 \in \mathrm{C}^d(C) \cap \mathrm{C_M}(e_1) \subset \mathrm{C}^a(C)$, $d \in$

$\mathrm{C}^a(C) \cap a \subset \mathcal{W}$.
Otherwise it follows $\tilde{d} = d = \psi_\kappa e_1$, $\kappa = \psi_{\mathrm{M}} d'$, $\kappa, e_1 \in \mathrm{C}^d(C) \cap \mathrm{C}_\kappa(e_1)$. If $a \prec \kappa$, it follows $\kappa, e_1 \in \mathrm{C}^d(C) \cap \mathrm{C}_\kappa(e_1) \subset \mathrm{C}^a(C)$, $d \in \mathrm{C}^a(C) \cap a \subset \mathcal{W}$. In the case $\kappa = a$ we have $a \in \mathsf{I}$, contradicting the assumption. If $\kappa \prec a$, it follows $\kappa = \psi_{\mathrm{M}} d' \in \mathrm{C}^\kappa(C)$, $d' \in \mathrm{C}^\kappa(C) \cap \mathrm{C}_{\mathrm{M}}(d') \subset \mathrm{C}^a(C)$ by Lemma 5.1 (a), $\kappa \in \mathrm{C}^a(C) \cap a \subset \mathcal{W}$.
In all cases we have therefore $d \preceq e \in \mathcal{W}$, $\mathcal{W} \cap e \cong W(\mathcal{W}) \cap e$, $C|d \cong \mathcal{W}|d \cong W(\mathcal{W})|d \cong W(C)|d$, $d \in W(C)|d \subset C$ and the induction is complete.
Let $C' := C \cup \{a\}$. $W(C') \cap a \cong W(C) \cap a \cong C' \cap a$. $\tau^{C'}(a) \cong \tau^{\mathcal{W}}(a) \subset \mathcal{W} \cap a \cong C \cong C' \cap a \subset W(C')$, $a \in \mathrm{M}(\mathcal{W})|a \cong \mathrm{M}(C')|a$, therefore $a \in W(C') \cap C'$, $W(C')|a \cong C'|a \cong C'$, $\mathrm{Ag}(C')$, $a \in C'$. With $C'' := \gamma_{\mathrm{V}}(C')$ we have $C'' : \mathcal{P}(\mathrm{N})$ (provable as in 4.9 (b)) $C''^{\mathrm{V,Cl}} \cong C'$, therefore $\mathrm{Ag}(C''^{\mathrm{V,Cl}})$ and $a \in C''^{\mathrm{V,Cl}} \subset \mathcal{W}$.
Lemma 5.4 $\forall x \in \mathcal{W}.\Omega_x \in \mathcal{W}$.
Proof: As in [Se98], Lemma 4.31.
Lemma 5.5 *Assume* $\psi_{\psi_{\mathrm{M}} a} b \in \mathcal{A}^{\mathcal{W}}(\mathcal{W})$, $Z := \mathrm{C}^{\psi_{\psi_{\mathrm{M}} a} b}(\mathcal{W}) \cap \mathrm{C}_{\psi_{\mathrm{M}} a} b$.

(a) $\mathrm{sc}_{\mathrm{M}}(a) \cup \{0, \mathrm{M}\} \subset Z$.

(b) If $x_1, x_2 \in Z, x =_{\mathrm{NF}} x_1 + x_2 \vee x =_{\mathrm{NF}} \varphi_{x_1} x_2 \vee x =_{\mathrm{NF}} \Omega_{x_1}$, *then* $x \in Z$.

(c) If $\kappa, c \in Z$, $\kappa \in \mathsf{R}$, $\kappa, c \in \mathrm{C}_\kappa(c)$, $\psi_{\mathrm{M}} a \preceq \kappa$, $c \prec b$, *then* $\psi_\kappa c \in Z$.

(d) If $c \in Z \cap \psi_{\mathrm{M}} a$, *then there exists* $A : \mathcal{P}_{\mathrm{V}}(\mathrm{N})$ *such that* $\mathrm{Ag}(A^{\mathrm{V,Cl}}) \wedge c \in Z \subset c + 1 \wedge A^{\mathrm{V,Cl}} \subset Z$.

Proof: only (a): $\psi_{\mathrm{M}} a \in \mathrm{C}_{\psi_{\mathrm{M}} a}(b)$, therefore $a \in \mathrm{C}_{\psi_{\mathrm{M}} a}(b)$, $\mathrm{sc}_{\mathrm{M}}(a) \subset \mathrm{C}_{\psi_{\mathrm{M}} a}(b)$, $\mathrm{sc}_{\mathrm{M}}(a) \prec \psi_{\mathrm{M}} a$, further $\psi_{\psi_{\mathrm{M}} a} b \in \mathrm{C}^{\psi_{\psi_{\mathrm{M}} a} b}(\mathcal{W})$, $\psi_{\mathrm{M}} a \in \mathrm{C}^{\psi_{\psi_{\mathrm{M}} a} b}(\mathcal{W})$, $a \in \mathrm{C}^{\psi_{\psi_{\mathrm{M}} a} b}(\mathcal{W})$, $\mathrm{sc}_{\mathrm{M}}(a) \subset \mathrm{C}^{\psi_{\psi_{\mathrm{M}} a} b}(\mathcal{W})$.
Lemma 5.6 *Assume* $\psi_{\psi_{\mathrm{M}} a} b \in \mathcal{A}^{\mathcal{W}}(\mathcal{W})$, $Z := \mathrm{C}^{\psi_{\psi_{\mathrm{M}} a} b}(\mathcal{W}) \cap \mathrm{C}_{\psi_{\mathrm{M}} a}(b)$. *Let* $Y_0^0 := \mathrm{sc}_{\mathrm{M}}(a) \cup \{0, \mathrm{M}\}$, P^0 *be a proof of* $Y^0 \subset Z$. *Assume* $Y^n : \mathcal{P}_{\mathrm{V}}(\mathrm{N})$ *be defined,* $P^n : (Y^n)^{\mathrm{V,Cl}} \subset Z$.
$Y_0^{n+1} := \{0, \mathrm{M}\} \cup \mathrm{sc}_{\mathrm{M}}(a) \cup$
$\{x \in \mathsf{OT} \mid \exists y, z \in (Y^n)^{\mathrm{V,Cl}}.x =_{\mathrm{NF}} y + z \vee x =_{\mathrm{NF}} \varphi_y z \vee x =_{\mathrm{NF}} \Omega_y\}$
$\cup \{x : \mathrm{N} \mid \exists \kappa \in \mathsf{R} \cap Z.\exists c \in Z.x = \psi_\kappa c \in \mathsf{OT} \wedge \psi_{\mathrm{M}} a \preceq \kappa \wedge c \prec b\}$
$\cup \{x : \mathrm{N} | \exists y \in (Y^n)^{\mathrm{V,Cl}} \cap \psi_{\psi_{\mathrm{M}} a} b.x \in (A_{y,P^n})^{\mathrm{V,Cl}}\})$,
where $A_{y,P^n} : \mathcal{P}_{\mathrm{V}}(\mathrm{N})$, $\mathrm{Ag}(A_{y,P^n}^{\mathrm{V,Cl}})$, $y \in A_{y,P^n}^{\mathrm{V,Cl}} \subset y + 1$ *(which in fact depends on the proof of* $y \in (Y^n)^{\mathrm{V,Cl}} \cap \psi_{\psi_{\mathrm{M}} a} b$*) and* $Y^n := \gamma_{\mathrm{V}}(Y_0^n)$, $Y^n : \mathcal{P}_{\mathrm{V}}(\mathrm{N})$. P_0^{n+1} *is a proof* $Y_0^{n+1} \subset Z$.
Let $Y_0 := \{y : \mathrm{N} \mid \exists n : \mathrm{N}.y \in Y_0^n\}$, $Y := \gamma_{\mathrm{V}}(Y_0)$.
Then it follows $Y : \mathcal{P}_{\mathrm{V}}(\mathrm{N})$, $Y^{\mathrm{V,Cl}} \cong Z$.
Proof:
We easily see that$Y^{\mathrm{V,Cl}} \cong Y_0$, $\forall n.(Y^n)^{\mathrm{V,Cl}} \cong Y_0^n$. Therefore it suffices to show $Y_0 \cong Z$.
"$\subset$" follows by 5.5.
"$\supset$" We first show by an easy induction on $\mathrm{length}(x)$:

$$(*) \qquad \forall x \in \mathsf{OT}.\mathrm{sc}_{\mathrm{M}}(x) \subset Y_0 \wedge \mathrm{G}_{\mathrm{M}}(x) \prec b \to x \in Y_0$$

Now we show by induction on $\mathrm{length}(a)$, $c \in Z \to c \in Y_0$:
If $c = 0, \mathrm{M}$, it follows $c \in Y_0$.
If $c \preceq \mathrm{sc}_{\mathrm{M}}(a)$, it follows $c \in Z| \max(\mathrm{sc}_{\mathrm{M}}(a))$, $\max(\mathrm{sc}_{\mathrm{M}}(a)) \in Y_0$, $c \in A_{\max(\mathrm{sc}_{\mathrm{M}}(a)),P^0}^{\mathrm{V,Cl}} \subset Y_0$.
If $c =_{\mathrm{NF}} a_1 + a_2, \varphi_{a_1} a_2, \Omega_{a_1}$, the assertion follows by IH.
Assume now $c = \psi_\kappa e$, $\max(\mathrm{sc}_{\mathrm{M}}(a)) \prec c$.
If $\psi_{\mathrm{M}} a \preceq c$, it follows $\kappa, e \in Z$, $e \prec b$, by IH the assertion.
If $c \prec \psi_{\mathrm{M}} a$, it follows $c \prec \psi_{\psi_{\mathrm{M}} a} b$, $c \in \mathcal{W}$.
Subcase $\kappa = \Omega_{e+1}$: $\Omega_e \in \mathcal{W}$, $e \in \mathcal{W}$, by IH $e \in Y_0$, $\kappa \in Y_0$, $c \in \mathcal{W} \cap \kappa \subset Y_0$.
Subcase $\kappa = \mathrm{M}$: then by $\psi_{\mathrm{M}} e \prec \psi_{\mathrm{M}} a$, it follows $e \prec a$, further $\psi_{\mathrm{M}} e \in \mathcal{W}$, $\psi_{\mathrm{M}} e \in \mathrm{C}^{\psi_{\mathrm{M}} e}(\mathcal{W})$,

$e \in \mathrm{C}^{\psi_{\mathrm{M}}e}(\mathcal{W})$, $e \in \mathrm{C}_{\mathrm{M}}(e)$, $\mathrm{sc}_{\mathrm{M}}(e) \subset \mathrm{C}_{\mathrm{M}}(e) \cap \mathrm{M} = \psi_{\mathrm{M}}e$, $\mathrm{sc}_{\mathrm{M}}(e) \subset \mathrm{C}^{\psi_{\mathrm{M}}e}(\mathcal{W}) \cap \psi_{\mathrm{M}}e \subset \mathcal{W} \cap \psi_{\psi_{\mathrm{M}}a}b \subset Z$, by IH $\mathrm{sc}_{\mathrm{M}}(e) \subset Y_0$, further $\mathrm{G}_{\mathrm{M}}(e) \prec e \prec a \prec b$, and by $(*)$ $e \in Y_0$. $\psi_{\mathrm{M}}e \prec \psi_{\mathrm{M}}a$, $\psi_{\mathrm{M}}e \preceq \mathrm{sc}_{\mathrm{M}}(a) \vee (e \prec a \wedge \mathrm{sc}_{\mathrm{M}}(e) \prec \psi_{\mathrm{M}}a)$, by $\max(\mathrm{sc}_{\mathrm{M}}(a)) \prec c$, $e \in Y_0$, $e \prec a \prec b$, $c = \psi_{\mathrm{M}}e \in Y_0$.
Subcase $\kappa = \psi_{\mathrm{M}}d$: $\psi_{\psi_{\mathrm{M}}d}e \prec \psi_{\psi_{\mathrm{M}}a}b$. Then $\psi_{\mathrm{M}}d \preceq \psi_{\psi_{\mathrm{M}}a}b \vee (d = a \wedge e \prec b) \vee \psi_{\psi_{\mathrm{M}}d}e \prec \psi_{\mathrm{M}}a \prec \psi_{\mathrm{M}}d$.
Subsubcase $\psi_{\mathrm{M}}d \preceq \psi_{\psi_{\mathrm{M}}a}b$. Then $\psi_{\mathrm{M}}d \prec \psi_{\mathrm{M}}a \wedge \mathrm{sc}_{\mathrm{M}}(d) \prec \psi_{\psi_{\mathrm{M}}a}b$, therefore $(\psi_{\mathrm{M}}d \preceq \mathrm{sc}_{\mathrm{M}}(a) \vee d \prec a) \wedge \mathrm{sc}_{\mathrm{M}}(d) \prec \psi_{\psi_{\mathrm{M}}a}b$. If $\psi_{\mathrm{M}}d \preceq \mathrm{sc}_{\mathrm{M}}(a)$, it follows $\psi_{\psi_{\mathrm{M}}d}e \in \mathcal{W}|\max(\mathrm{sc}_{\mathrm{M}}(a)) \subset Y_0$. Otherwise $d \prec a \wedge \mathrm{sc}_{\mathrm{M}}(d) \prec \psi_{\psi_{\mathrm{M}}a}b$. $\psi_{\psi_{\mathrm{M}}d}e \in \mathcal{W}$, therefore $\psi_{\mathrm{M}}d \in \mathrm{C}^{\psi_{\psi_{\mathrm{M}}d}e}(\mathcal{W}) \cap \mathrm{C}_{\psi_{\mathrm{M}}d}(e) \subset \mathrm{C}^{\psi_{\mathrm{M}}d}(\mathcal{W})$ by Lemma 5.1 (a), $d \in \mathrm{C}^{\psi_{\mathrm{M}}d}(\mathcal{W})$, $\mathrm{sc}_{\mathrm{M}}(d) \subset \mathrm{C}^{\psi_{\mathrm{M}}d}(\mathcal{W})$, $\mathrm{sc}_{\mathrm{M}}(d) \subset \mathrm{C}_{\psi_{\mathrm{M}}d}(e) \cap \mathrm{C}_{\mathrm{M}}(d) \cap \mathrm{M} = \psi_{\psi_{\mathrm{M}}d}e$, $\mathrm{sc}_{\mathrm{M}}(d) \subset \mathrm{C}^{\psi_{\psi_{\mathrm{M}}d}e}(\mathcal{W} \cap \psi_{\psi_{\mathrm{M}}d}e \subset Z$. By IH $\mathrm{sc}_{\mathrm{M}}(d) \subset Y_0$, $\mathrm{G}_{\mathrm{M}}(d) \prec d \prec a$, by $(*)$ $d \in Y_0$. Since $d \prec a$, it follows $\psi_{\mathrm{M}}d \in Y_0$, $\psi_{\psi_{\mathrm{M}}d}e \in A^{\mathrm{V,Cl}}_{\psi_{\mathrm{M}}d,P^n} \subset Y_0$ for some $n : \mathrm{N}$.
Subsubcase $d = a \wedge e \prec b$: $c \in \mathcal{W}$. Therefore $e, \psi_{\mathrm{M}}a \in \mathrm{C}^{\psi_{\psi_{\mathrm{M}}a}e}(\mathcal{W}) \cap \mathrm{C}_{\psi_{\mathrm{M}}a}(e)$, by IH $\psi_{\mathrm{M}}a, e \in Z$, $e \prec b$, therefore $c \in Z$.
Subsubcase $\psi_\kappa e \prec \psi_{\mathrm{M}}a \prec \kappa$. From $\psi_\kappa e \prec \psi_{\mathrm{M}}a$ it follows $\psi_\kappa e \preceq \mathrm{sc}_{\mathrm{M}}(a)$, $\psi_\kappa e \in Y_0$.
Lemma 5.7 $a \in \mathcal{A}^{\mathcal{W}}(\mathcal{W}) \cap \mathrm{M} \Rightarrow \mathcal{A}^{\mathcal{W}}(\mathcal{W}) \cap a \cong \mathcal{W} \cap a$.
Proof:
"$\supset$": trivial.
"$\subset$": Assume $x \in \mathcal{A}^{\mathcal{W}}(\mathcal{W}) \cap a$, show $x \in \mathcal{W}$ by induction on length(x).
If $x = 0$, this is trivial. If $x =_{\mathrm{NF}} x_1 + x_2, \varphi_{x_1}x_2, \Omega_{x_1}$, it follows $x \in \mathrm{C}^x(\mathcal{W})$, $x_i \in \tau^{\mathcal{W}}(x) \subset \mathcal{W}$, $x \in \mathcal{W}$.
If $x = \psi_{\Omega_{c+1}}c$, it follows $\Omega_c \in \tau^{\mathcal{W}}(x) \subset \mathcal{W}$, $\Omega_{c+1} \in \mathcal{W}$, $x \in \mathcal{A}^{\mathcal{W}}(\mathcal{W}) \cap \Omega_{c+1} \cong \mathcal{A}^{\mathcal{W}}(W(\mathcal{W})) \cap \Omega_{c+1} \subset W(\mathcal{W}) \cap \Omega_{c+1} \cong \mathcal{W} \cap \Omega_{c+1}$.
If $x = \psi_{\mathrm{M}}b$, it follows $\psi_{\mathrm{M}}b \in \mathrm{C}^{\psi_{\mathrm{M}}b}(\mathcal{W})$, by 5.1 (a) $b \in \mathrm{C}^{\psi_{\mathrm{M}}b}(\mathcal{W}) \cap \mathrm{C}_{\mathrm{M}}(b) \subset \mathrm{C}^a(\mathcal{W})$, $\psi_{\mathrm{M}}b \in \mathrm{C}^a(\mathcal{W}) \cap a \subset \mathcal{W}$.
Case $x = \psi_{\psi_{\mathrm{M}}b}c$. By Lemma 5.6 exists a $Y : \mathcal{P}_{\mathrm{V}}(\mathrm{N})$ such that $Y^{\mathrm{V,Cl}} \cong \tau^{\mathcal{W}}(\psi_{\psi_{\mathrm{M}}b}c)$. Then by Lemma 5.3 it follows $\psi_{\psi_{\mathrm{M}}a}b \in \mathcal{W}$.

From now on, we will work in MLM.
Lemma 5.8 *(a) If $a \in \mathrm{C}^b(B)$, then $\exists C : \mathrm{N}.C \in \mathcal{P}^{fin}(\mathrm{N}) \wedge a \in \mathrm{C}^b(\mathrm{C}^{fin,\mathrm{Cl}}) \wedge \mathrm{C}^{fin,\mathrm{Cl}} \subset B$, where $\mathrm{C}^{fin,\mathrm{Cl}} := \{y : \mathrm{N} \mid \mathrm{atom}(y \in_{fin} C)\}$.*

(b) $a \in \mathrm{C}^b(B)$, $B \subset C \Rightarrow a \in \mathrm{C}^b(C)$.

(c) Assume $C : \mathcal{P}_{\mathrm{V}}(\mathrm{N})$, $C \subset \mathcal{W}$. Then $\exists A : \mathcal{P}_{\mathrm{V}}(\mathrm{N}).\mathrm{Ag}(A^{\mathrm{V,Cl}}) \wedge C \subset A^{\mathrm{V,Cl}}$.

(d) $\forall x \in \mathrm{C}^a(\mathcal{W}).\exists C : \mathcal{P}_{\mathrm{V}}(\mathrm{N}).\mathrm{Ag}(\mathrm{C}^{\mathrm{V,Cl}}) \wedge a \in \mathrm{C}^a(\mathrm{C}^{\mathrm{V,Cl}}))$.

(e) (a) - (d) can be proved in ML_2.
Proof: (a): Induction on length(a). (b): easy, (c): by [Se98], Lemma 4.26.
Definition 5.9 *(a)* $\mathrm{Univ}_0(u_0, v_0) := u_0$, $\mathrm{Univ}_1(u_0, v_0, z_1) := v_0 z_1$, $\mathrm{Univ}(u_0, v_0) := (x_1, y_1)\mathrm{Univ}_0(u_0, v_0), (x_1, y_1, z_1)\mathrm{Univ}_1(u_0, v_0, z_1)$. *Under the assumption $u_0 : \mathrm{V}$, $v_0 : \mathrm{T}(u_0) \to \mathrm{V}$ we have* $x_1 : \mathrm{V}, y_1 : \mathrm{T}(u) \to \mathrm{V} \Rightarrow \mathrm{Univ}_0(u_0, v_0) : \mathrm{V}$, $x_1 : \mathrm{V}, y_1 : \mathrm{T}(u) \to \mathrm{V}, z_1 : \mathrm{T}(\mathrm{Univ}_0(u_0, v_0)) \Rightarrow \mathrm{Univ}_1(u_0, v_0, z_1) : \mathrm{V}$.

(b) Let $\vec{f} := \mathrm{Univ}(u_0, v_0)$.
$u_1 := \widehat{\mathrm{N}}_{\vec{f}}$, $v_1 := (x)\widehat{\mathrm{N}}_{0,\vec{f}}$, $\iota_{0,\vec{f}} := \mathrm{Res0}_{\vec{f}}(u_1, v_1)$, $\iota_{1,\vec{f}}(y) := \mathrm{Res1}_{\vec{f}}(u_1, v_1, y)$. *Then, assuming u_0, v_0 as above we have* $\mathrm{s}_{\vec{f}}(\iota_{0,\vec{f}}) = \mathrm{Univ}_0(u_0, v_0) = u_0$, *and for* $x : \mathrm{T}_{\vec{f}}(\iota_{0,\vec{f}})$ $\mathrm{s}_{\vec{f}}(\iota_{1,\vec{f}}(y)) = \mathrm{Univ}_1(u_0, v_0, y) = v_0 y$.

Lemma 5.10 *Assume $a \in \mathcal{A}^{\mathcal{W}}(\mathcal{W})$. Then there exists f, g such that $u_0 : \mathrm{V}, v_0 : \mathrm{T}(u_0) \to \mathrm{V} \Rightarrow f(u_0, v_0) : \mathrm{V}$, $u_0 : \mathrm{V}, v_0 : \mathrm{T}(u_0) \to \mathrm{V}, z_1 : \mathrm{T}(f(u_0, v_0)) \Rightarrow g(u_0, v_0, z_1) : \mathrm{V}$, and with $f := (u_0, v_0)s_0$, $g := (u_0, v_0, z_1)t_0$, $\vec{f} := f, g$ we have $a \in \mathcal{A}^{\mathcal{W}_{\vec{f}}^{\mathrm{V,Cl}}}(\mathcal{W}_{\vec{f}}^{\mathrm{V,Cl}})$*

Proof:

$a \in \mathrm{C}^a(\mathcal{W})$. By Lemma 5.8 (d) exists $C : \mathcal{P}_\mathrm{V}(\mathrm{N})$ such that $\mathrm{Ag}(\mathrm{C}^{\mathrm{V,Cl}})$ and $a \in \mathrm{C}^a(\mathrm{C}^{\mathrm{V,Cl}})$. Let $C' := \mathrm{C}^{\mathrm{V,Cl}}$. Assume $u_0 : \mathrm{V}$, $v_0 : \mathrm{T}(u_0) \to \mathrm{V}$. Let $D := \mathcal{W}^{\mathrm{V,Cl}}_{\mathrm{Univ}(u_0,v_0)} \cup C'$. $\mathrm{Ag}(D)$, $\gamma_\mathrm{V}(D) : \mathcal{P}_\mathrm{V}(D)$. $\mathrm{C}^a(D) \cap a \subset \tau^{\mathcal{W}}(a) \subset \mathcal{W}$. Therefore there exists $E_{u_0,v_0} : \mathcal{P}_\mathrm{V}(\mathrm{N})$, $\mathrm{Ag}(E_{u_0,v_0})$ such that $\mathrm{C}^a(D) \cap a \subset E^{\mathrm{V,Cl}}_{u_0,v_0}$. $f := (x_1, y_1)\widehat{\mathrm{N}}_\mathrm{V}$, and $g := (x_1, y_1, z_1)z_1 \in_\mathrm{V} E_{u_0,v_0}$

We have now:

(i) $a \in \mathrm{C}^a(\mathcal{W}_{\vec{f}}^{\mathrm{V,Cl}})$:

$u_1 := \widehat{\mathrm{N}}_{0,\vec{f}} : \mathrm{U}_{\vec{f}}$, $u_1' := \mathrm{s}_{\vec{f}}(u_1)$, $v_1 := (x)\mathrm{C}_0(x)$, $v_1' := \lambda x.\mathrm{s}_{\vec{f}}(\mathrm{C}_0(x))$, $u_2 := \mathrm{Res0}_{\vec{f}}(u_1, v_1) : \mathrm{U}_{\vec{f}}$, $E_0 := \lambda y.\mathrm{Res1}_{\vec{f}}(u_1, v_1, y) : \mathcal{P}_{\vec{f}}(\mathrm{N})$, $\mathrm{s}_{\vec{f}}(u_2) = f(u_1', v_1') = \widehat{\mathrm{N}}_\mathrm{V}$, and for $x : \mathrm{T}(\widehat{\mathrm{N}}_\mathrm{V}) = \mathrm{N}$, $\mathrm{s}_{\vec{f}}(x \in_{\vec{f}} E_0) = \mathrm{s}_{\vec{f}}(\mathrm{Res1}_{\vec{f}}(u_1, v_1, x)) = x \in_\mathrm{V} E_{u_1',v_1'}$, as above $C' \cap a \subset D \cap a \subset \mathrm{C}^a(D) \cap a \subset E^{\mathrm{V,Cl}}_{u_1',v_1'}$, $E^{\mathrm{V,Cl}}_{u_1',v_1'} \cong E_0^{\vec{f},\mathrm{Cl}}$, $\mathrm{Ag}(E^{\mathrm{V,Cl}}_{u_1',v_1'})$, therefore $\mathrm{Ag}(E_0^{\vec{f},\mathrm{Cl}})$, $a \in \mathrm{C}^a(C')$, $a \in \mathrm{C}^a(C' \cap a)$, $a \in \mathrm{C}^a(E_0^{\vec{f},\mathrm{Cl}}) \subset \mathrm{C}^a(\mathcal{W}_{\vec{f}}^{\mathrm{V,Cl}})$.

(ii) $\mathrm{C}^a(\mathcal{W}_{\vec{f}}^{\mathrm{V,Cl}}) \cap a \subset \mathcal{W}_{\vec{f}}$:

Assume $x \in \mathrm{C}^a(\mathcal{W}_{\vec{f}}^{\mathrm{V,Cl}}) \cap a$. Then $x \in \mathrm{C}^a(A^{\vec{f},\mathrm{Cl}}) \cap a$ for some $A : \mathcal{P}_{\vec{f}}(\mathrm{N})$, $\mathrm{Ag}(A^{\vec{f},\mathrm{Cl}})$. Let $A' := A^{\vec{f},\mathrm{Cl}}$, $u_2 := \widehat{\mathrm{N}}_{\vec{f}}$, $u_2' := \mathrm{s}_{\vec{f}}(\widehat{\mathrm{N}}_{\vec{f}})$, $v_2 := (y)y \in_{\vec{f}} A$, $v_2' := \lambda y.\mathrm{s}_{\vec{f}}(y \in_{\vec{f}} A)$, $E_0 := \lambda x.\mathrm{Res1}_{\vec{f}}(u_2, v_2, x)$. For $x : \mathrm{N}$ we have $\mathrm{s}_{\vec{f}}(x \in_{\vec{f}} E_0) = g(u_2', v_2', x) = x \in_\mathrm{V} E_{u_2',v_2'}$, therefore $E_0^{\vec{f},\mathrm{Cl}} \cong E^{\mathrm{V,Cl}}_{u_2',v_2'}$. $\mathrm{C}^a(\mathcal{W}_{\mathrm{Univ}(u_2',v_2')^{\mathrm{V,Cl}}} \cup C') \cap a \subset E^{\mathrm{V,Cl}}_{u_2',v_2'} \cong E_0^{\vec{f},\mathrm{Cl}}$, $\mathrm{Ag}(E^{\mathrm{V,Cl}}_{u_2',v_2'})$, $\mathrm{Ag}(E_0^{\vec{f},\mathrm{Cl}})$. By Definition 5.9b $\iota_{0,u_2',v_2'} : \mathrm{U}_{\mathrm{Univ}(u_2',v_2')}$, $\mathrm{s}_{\mathrm{Univ}(u_2',v_2')}(\iota_{0,u_2',v_2'}) = u_2' = \widehat{\mathrm{N}}_\mathrm{V}$, $\mathrm{s}_{\mathrm{Univ}(u_2',v_2')}(\iota_{1,u_2',v_2'}(y)) = v_2'y = \mathrm{s}_{\vec{f}}(y \in_{\vec{f}} A)$, with $B := \lambda y.\iota_{1,u_2',v_2'}(y)$ we have $B^{\mathrm{Univ}(u_2',v_2'),\mathrm{Cl}} \cong A^{\vec{f},\mathrm{Cl}}$, by $\mathrm{Ag}(A^{\vec{f},\mathrm{Cl}})$ $\mathrm{Ag}(B^{\mathrm{Univ}(u_2',v_2'),\mathrm{Cl}})$, $A^{\vec{f},\mathrm{Cl}} \cong B^{\mathrm{Univ}(u_2',v_2'),\mathrm{Cl}} \subset$
$\mathcal{W}_{\mathrm{Univ}(u_2',v_2')}$, $x \in \mathrm{C}^a(A^{\vec{f},\mathrm{Cl}}) \cap a \subset \mathrm{C}^a(\mathcal{W}_{\mathrm{Univ}(u_2',v_2')} \cup C') \cap a \subset E^{\mathrm{V,Cl}}_{u_2',v_2'} \cong E_0^{\vec{f},\mathrm{Cl}} \subset \mathcal{W}_{\vec{f}}^{\mathrm{V,Cl}}$.

Lemma 5.11 *(a) $\mathcal{A}^{\mathcal{W}}(\mathcal{W}) \cap \mathrm{M} \subset \mathcal{W}$.*

(b) $\mathcal{W} \cap \mathrm{M} \cong \mathrm{W}(\mathcal{W}) \cap \mathrm{M}$.

Proof:

(a) Assume $a \in \mathcal{A}^{\mathcal{W}}(\mathcal{W}) \cap \mathrm{M}$. By 5.10 exists $\vec{f}$ as before such that $a \in \mathcal{A}^{\mathcal{W}_{\vec{f}}^{\mathrm{V,Cl}}}(\mathcal{W}_{\vec{f}}^{\mathrm{V,Cl}})$. Let $\mathcal{W}' := \mathcal{W}_{\vec{f}}^{\mathrm{V,Cl}}$. By 5.7 it follows $\mathcal{A}^{\mathcal{W}'}(\mathcal{W}') \cap a \cong \mathcal{W}' \cap a$. Therefore $\mathrm{W}(\mathcal{W}') \cap a \cong \mathcal{W}' \cap a$. $A := (\mathcal{W}' \cap a) \cup \{a\}$. $\mathrm{Ag}(A)$, $\gamma_\mathrm{V}(A) : \mathcal{P}(\mathrm{N})$, $\gamma_\mathrm{V}(A)^{\mathrm{V,Cl}} \cong A$, $a \in A \subset \mathcal{W}$.

(b): $\mathcal{W} \subset \mathrm{W}(\mathcal{W})$, and by $\mathrm{Ind}(y \in \mathrm{W}(\mathcal{W}))$, using (a) it follows $\forall y \in \mathrm{W}(\mathcal{W}).y \prec \mathrm{M} \to y \in \mathcal{W}$.

Definition 5.12 $\mathcal{W}_0 := \mathcal{W} \cap \mathrm{M}$, $\mathcal{W}_{\mathrm{S}(i)} := \mathrm{W}(\mathcal{W}_i) \cap \Omega_{\mathrm{M}+1\cdot\mathrm{S}(i)}$.

In the following we write $\mathrm{M}+i$ instead of $\mathrm{M}+1\cdot i$, similar for $j, \mathrm{S}(j), \mathrm{S}(i)$ etc. instead of i.

Lemma 5.13 *For all $i \prec \omega$ $\mathrm{Ag}(\mathcal{W}_i) \wedge \Omega_{\mathrm{M}+i} \in \mathcal{W}_{\mathrm{S}(i)} \wedge \mathcal{W}_i \cong \mathcal{W}_{\mathrm{S}(i)} \cap \Omega_{\mathrm{M}+i}$.*

Proof: As in [Se98], Lemma 4.40.

Theorem 5.14 *For all $n \in \mathbb{N}$ we have:*

(a) $\mathrm{MLM} \vdash \forall X : \mathcal{P}(\mathrm{N}).(\forall y \in \mathsf{OT}.(\forall x \prec y.x \in X) \to y \in X) \to \forall y \prec \psi_{\Omega_1}\Omega_{\mathrm{M}+n}.y \in X$.

(b) If $\mathrm{MLM} \vdash \Delta \Rightarrow A \in \mathrm{Cl}(\mathrm{N})$, *then* $\mathrm{MLM} \vdash \Delta \Rightarrow (\forall y \in \mathsf{OT}.(\forall x \prec y.x \in A) \to y \in A) \to \forall y \prec \psi_{\Omega_1}\Omega_{\mathrm{M}+n}.y \in A$.

Proof: As Theorem 4.41 in [Se98]

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