

# Ordinal Systems, Part 2: One Inaccessible

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**Abstract.** We develop an alternative approach to well-ordering proofs beyond the Bachmann-Howard ordinal using transfinite sequences of ordinal notations and use it in order to carry out well-ordering proofs for  $\sigma$ -ordinal systems. We extend the approach of ordinal systems as an alternative way of presenting ordinal notation systems started in [18] and develop ordinal systems, which have in the limit exactly the strength of Kripke-Platek set theory with one recursively inaccessible. The upper bound is determined by giving well-ordering proofs, which use the technique of transfinite sequences. We derive from the new approach the traditional approach to well-ordering proofs using distinguished sets. The lower bound is determined by extending the concept of ordinal function generators in [18] to inaccessibles.

## 1 Introduction

This article is a followup of [18]. In that article we introduced ordinal systems as an alternative way of describing ordinal notation systems which usually make use of collapsing functions. In the simplest case of one single ordinal system, ordinals were always denoted by referring to smaller ones, a principle, which is valid in the case of systems built from extensions of the Veblen function. In order to get beyond the Bachmann Howard ordinal, we used several ordinal systems in parallel, such that the ordinal systems refer to each other in a controlled way. We described, how the combination of ordinal systems was used in order to generate in a well-ordered way the ordinal notation systems in question. This showed that they are well-ordered. We considered and analyzed ordinal systems up to the proof theoretic strength of  $ID_\sigma$ .

In the current article we proceed with this research and present I-ordinal systems: ordinal systems which reach in the limit the strength of KPI, Kripke Platek set theory with one recursively inaccessible I (more precisely, the class of ordinals in KPI has the properties of one recursively inaccessible; proof theoretic studies for variations of Kripke Platek set theory can be found in [9], a formalization of KPI can be found in [9] or looked up as well in [4]). We will then determine their strength as that of KPI. This presentation should help to understand ordinal notation systems of this strength in a better way.

The content of the article is as follows: I-ordinal systems will be introduced in Section 3. In Sections 4 and 5 we will then show that I-ordinal systems are well-ordered and that elementary I-ordinal systems, i.e. I-ordinal systems, the properties of which can be verified in PRA, have order type less than KPI. We are going to present two well-ordering proofs. One, using a new technique, namely transfinite sequences of ordinal notations, and a second in the more traditional way, which uses distinguished sets and can be used for well-ordering proofs in constructive theories. In order to motivate the new technique of using sequences, we introduce it in Section 2 in the context of  $\sigma$ -ordinal systems. In the last section of this article (Section 6), we will extend the notion of ordinal function generators and define ordinal function generators corresponding to I-OS. We will see there that we reach the strength of I-OS. Therefore the strength of elementary I-ordinal systems is determined as being precisely the strength of KPI.

## 2 A variation of the usual well-ordering proof techniques

The usual technique in proof theory for carrying out well-ordering proofs is the technique of distinguished sets, introduced by Buchholz in [1]. In our previous article [18] some (but not the full power of it) was used and it seems to be still the only technique available for carrying out well-ordering proofs of strength beyond  $ID_\omega$  in constructive theories. However, we believe that for theories which have substantially stronger strength it is not so easy to understand what is going on in it, and found it very difficult to teach it. We have now developed a variation of it, which we hope is slightly more intuitive. The disadvantage of it is that it can in the current form not be carried out in constructive theories. However this is no harm since from it in a second step we can derive the technique of distinguished sets. We hope our presentation motivates distinguished sets in a better way.

Instead of defining the sets in an abstract way we are going to define by recursion on set theoretic ordinals a transfinite sequence of ordinal notations  $a_\alpha$ . Distinguished sets will occur as initial segments of the set of notations generated in such a way. In order to show the well-foundedness of the resulting system, apart from some verifications, which can be essentially be carried out in primitive recursive arithmetic, and the existence of such a sequence, which is clear in the case of Kripke Platek set theory — the use of classical logic is crucial here —, we will need that, when iterated over all ordinals, the resulting sequence is complete, i.e. this sequence can not be extended.

In order to prove the consistency of the theories in question we will need therefore essentially that the class of ordinals is big enough. One can use principles which claim that the class of ordinals fulfills the properties of some recursively large ordinal. Or one can claim instead that the set of ordinals should be unbounded in the sense that we cannot exhaust it by any recursive process, and that therefore such an ordinal must exist. The latter point of view provides an alternative argument for the consistency of the theory in question, alternative to the use of

constructive theories, the consistency of which can be philosophically shown or at least justified. At the moment, the new argument is not as well philosophically developed as the other approach, but we hope to be able to investigate it in the near future. In this article, we only mention the possibility of such an approach. Once we have found such a sequence and carried out the well-ordering proof using it, we can now find the constructive version of the well-ordering proof using distinguished sets by formalizing abstractly the principles according to which the sets  $A_\alpha := \{a_\beta \mid a_\beta \downarrow \wedge \beta < \alpha\}$  are defined.

We will apply this in section 4 to I-ordinal systems. In a first step we want to look at  $\sigma$ -ordinal systems in order to introduce the technique. In the case of  $\sigma$ -ordinal systems, the benefits will not be very big, most readers will certainly prefer the old technique. However for stronger systems this technique will motivate well-ordering proofs in a better way.

We need a theory of the strength of  $ID_\sigma$  in which we have ordinals available. Kripke Platek set theory without  $\Delta_0$ -collection, but extended by natural numbers and the existence of a well-ordering  $\Sigma$  (as new urelemente) of order type  $\sigma$  and for every  $a \in \Sigma$  an admissible such that these admissibles are ordered as the corresponding elements of  $\Sigma$  and the collection of predecessors of an  $a \in \Sigma$  is a set, which is an element of the  $a$ th admissible, should be the correct theory. (The exact formulation of such a theory is not important here since this Section mainly serves to introduce and motivate the new technique). In this section, (set-theoretic) ordinal refers to the set of ordinals below some of the admissibles which were additionally introduced in this theory.

Before defining such sequences we make our life easier by extending the order  $<_\xi$  to a global order:

- Definition 2.1.** (a)  $T := \bigcup_{\xi \in \Sigma} T_\xi$ ,  $T_{<\xi} := \bigcup_{\mu < \xi} T_\mu$ ,  $T_{\leq \xi} := \bigcup_{\mu \leq \xi} T_\mu$ .  
(b) If  $a \in T_\xi$ ,  $b \in T_\mu$ , then  $a \prec b \Leftrightarrow (\xi < \mu \vee (\xi = \mu \wedge a \prec_\xi b))$ .  
(c) If  $a \in T_\xi$ , then  $k(a) := \bigcup_{\mu \in \Sigma} k_{\xi, \mu}(a)$ .  
(d) If  $A \subseteq T$ ,  $a \in T$ ,  $A \cap a := \{b \in A \mid b \prec a\}$

It seems, as if we are back to the traditional way of ordinal notation systems, in which small ordinals (e.g.  $a \in T_0$ ) are denoted by referring to bigger ones (e.g.  $b \in k(a) \cap T_1$ , therefore  $a \prec b$ ). But the real picture to keep in mind should be that of several ordinal systems, which refer to each other in a controlled way, i.e. smaller ordinal systems refer to bigger ordinal systems restricted to smaller levels only. The global order will be helpful in order to reduce the amount of syntax.

In the well-ordering proof we defined by recursion on  $\xi \in \Sigma$  sets  $M_\xi, Acc_\xi \subseteq T_\xi$  by:

$$\begin{aligned} M_\xi &:= T_\xi \upharpoonright_{\mu < \xi} Acc_\mu, \\ Acc_\xi &:= Acc_{\prec}(M_\xi). \end{aligned}$$

We can now use transfinite sequences of ordinal notations in order to obtain the same sets. We proceed as follows: we consider in increasing order all elements of

$\Sigma$ . For each  $\xi \in \Sigma$  we select, as long as possible, successively the least element of  $T_\xi$  which has smaller components in the set of ordinals already selected before, but has not been selected itself. Once we cannot find any more such a minimal element, we select the next element of  $\Sigma$ . Once we have finished, the elements of  $T$  chosen will be  $\bigcup_{\xi \in \Sigma} \text{Acc}_\xi$ .

In order to formalize this idea, we first define for sets  $A \subseteq T$   $M[A] := \{a \in T \mid k(a) \cap a \subseteq A\}$  and  $M_\xi[A] := M[A] \cap T_\xi$ . Now we define by recursion on  $\gamma$  elements  $a_\gamma \in T$  together with  $i_\gamma \in \Sigma$ ,  $A_\gamma \subset T_\gamma$  as follows:

$$\begin{aligned} A_\gamma &:= \{a_\beta \mid \beta < \gamma \wedge a_\beta \downarrow\}, \\ i_\gamma &:= \simeq \min_{<} \{\xi \in \Sigma \mid \min_{<}(M_\xi[A_\gamma] \setminus A_\gamma) \downarrow\}, \\ a_\gamma &:= \simeq \min_{<}(M_{i_\gamma}[A_\gamma] \setminus A_\gamma). \end{aligned}$$

**Convention 2.2.** A simultaneous definition of sequences like  $a_\gamma$ ,  $i_\gamma$  above should in the following be understood as follows: both  $i_\gamma$ ,  $a_\gamma$  (and possibly other sequences) are defined by recursion on the (set theoretic) ordinals  $\gamma$  and can have either value as indicated (above in  $\Sigma$  and  $T$ , respectively) or a symbol for undefined.  $a \downarrow$  means that  $a$  is not a symbol for undefined,  $a \uparrow$  means that  $a$  is (a symbol for) undefined. Similarly, for a set  $A$   $\min_{<}(A) \downarrow$  ( $\min_{<}(A) \uparrow$ ) means that a minimum of  $A$  with respect to  $<$  exists (does not exist). For two possibly objects of a set with an undefined element  $a \simeq b : \Leftrightarrow (a \downarrow \vee b \downarrow) \rightarrow a = b$ .

In the above definition  $i_\gamma$  is defined iff  $\forall \beta < \alpha (i_\beta \downarrow \wedge a_\beta \downarrow)$  and a minimum as in the definition above exists. If  $i_\gamma$  is defined, it has as value the corresponding minimal element. The definition of  $a_\gamma$  is similar, but presupposes that  $i_\gamma$  is defined. Similar future definitions are to be understood in the same way.

Therefore we start to define  $a_\gamma$ ,  $i_\gamma$ , as long as possible. If this is no longer the case, both get the value undefined. Note that this can be defined in the classical set theory used, in which we can “decide” whether  $i_\gamma$ ,  $a_\gamma$  are defined. Because of this definition, this proof is non-constructive.

**Lemma 2.3. (Admissibles are  $\Pi_2$ -reflecting)**

Assume  $\phi_0(x)$ ,  $\phi_1(x, y)$  are  $\Sigma$ -formulas.

(a) (Provable in Kripke Platek set theory without  $\Delta_0$ -collection but with natural numbers.) If  $\kappa$  is an admissible ordinal, then from

$$\begin{aligned} &\exists \alpha < \kappa . \phi_0(\alpha) \text{ and} \\ &\forall \alpha < \kappa . \exists \beta < \kappa . \phi_1(\alpha, \beta) \end{aligned}$$

follows the existence of some  $\delta < \kappa$  such that

$$\begin{aligned} &\exists \alpha < \delta . \phi_0(\alpha) \text{ and} \\ &\forall \alpha < \delta . \exists \beta < \delta . \phi_1(\alpha, \beta). \end{aligned}$$

(b) In the presence of  $\Delta_0$ -collection (for the universe of sets) (a) holds for the class of ordinals, i.e.

$$\exists \alpha . \phi_0(\alpha) \text{ and}$$

$$\forall \alpha. \exists \beta. \phi_1(\alpha, \beta)$$

implies the existence of some  $\delta$  such that the conclusion of (a) holds.

(c) (a), (b) hold as well with  $\phi_0, \phi_1$  replaced by several  $\Sigma$ -formulas.

*Proof.* (a): Define  $\alpha_n$  for  $n \in \omega$  inductively by:

$\alpha_0 = \alpha + 1$ , where  $\alpha$  minimal such that  $\phi_0(\alpha)$  holds.  $n \rightarrow n + 1$ :  $\forall \beta < \alpha_n. \exists \gamma < \kappa. \phi_1(\beta, \gamma)$ . By  $\Sigma$ -collection there exists  $\rho < \kappa$  such that  $\forall \beta < \alpha_n. \exists \gamma < \rho. \phi_1(\beta, \gamma)$ . Choose  $\alpha_{n+1} = \max\{\rho, \alpha_n\}$  for the minimal such  $\rho$ . With  $\delta := \sup_{n \in \omega} \alpha_n < \alpha$  follows now the assertion.

(b) follows similarly and (c) by coding several  $\Sigma$ -formulas into one.

**Definition 2.4.** (a)  $A$  is  $T_\xi$ -complete iff

$$(\forall \mu \leq \xi. \forall a \in M_\mu[A](M_\mu[A] \cap a \subseteq A) \rightarrow a \in A) .$$

(b)  $A$  is complete iff  $A$  is  $T_\xi$ -complete for all  $\xi \in \Sigma$ .

**Remark 2.5.** (a)  $(\gamma < \gamma' \wedge i_{\gamma'} \downarrow) \Rightarrow i_\gamma \leq i'_\gamma$ .

(b)  $(\alpha < \beta \wedge i_\beta \downarrow \wedge i_\alpha = i_\beta) \Rightarrow a_\alpha \prec a_\beta$

(c)  $A_\beta$  are  $\prec$ -well-ordered.

(d)  $A_\beta$  is  $T_\xi$ -complete iff  $i_\beta \uparrow$  or  $\xi \leq i_\beta$ .

$A_\beta$  is complete iff  $a_\beta \uparrow$ .

(e) If  $A_\beta$  is  $T_\xi$ -complete (complete) and  $\beta < \gamma$ , then  $A_\gamma$  is  $T_\xi$ -complete (complete).

*Proof.* (a): easy induction on  $\gamma'$ .

(b): Induction on  $\beta$ .  $a_\beta \neq a_\alpha$ . Let  $\xi := i_\alpha$  and assume we had  $a_\beta \prec a_\alpha$ . By (a)  $A_\beta \cap T_{<\xi} = A_\alpha \cap T_{<\xi}$ . By  $a_\beta \prec a_\alpha$  it follows  $a_\beta \in M_\xi[A_\beta] \setminus M_\xi[A_\alpha]$ , there exists  $c \in k(a_\beta) \cap T_\xi$  such that  $c \in A_\beta \setminus A_\alpha$ ,  $c = a_\gamma$  for some  $\alpha < \gamma < \beta$ ,  $a_\gamma \prec a_\beta \prec a_\alpha$ , contradicting the IH.

(c): By (a), (b). (d) Immediate. (e) By (a), (d).

The following Lemma will not be needed for the well-ordering proof. We state it in order to show that we have defined exactly the sets  $\text{Acc}_\xi$  from the original well-ordering proof.

**Lemma 2.6.** (a) If we define  $i'_\gamma, a'_\gamma, A'_\gamma$  by

$$\begin{aligned} A'_\gamma &:= \{a'_\beta \mid \beta < \gamma \wedge a'_\beta \downarrow\}, \\ i'_\gamma &:= \min_{<} \{\xi \in \Sigma \mid \min_{<} \{a \in T_\xi \setminus A'_\gamma \mid k(a) \cap T_{<\xi} \subseteq A'_\gamma\} \downarrow\}, \\ a'_\gamma &:= \min_{<} \{a \in T_{i'_\gamma} \setminus A'_{i'_\gamma} \mid k(a) \cap T_{<i'_\gamma} \subseteq A'_\gamma\}, \end{aligned}$$

then  $i'_\gamma \simeq i_\gamma, a'_\gamma \simeq a_\gamma$  for all  $\gamma$ .

(b) If  $A_\gamma$  is  $T_\xi$ -complete, then  $A_\gamma = \text{Acc}_\xi$ , where  $\text{Acc}_\xi$  is defined as in [18].

*Proof.* (a) Induction on  $\gamma$ . By IH  $A'_\gamma = A_\gamma$ . Let for  $\xi \in \Sigma$ ,  $B_\xi := \{a \in T_\xi \setminus A_\gamma \mid k(a) \cap T_{<\xi} \subseteq A_\gamma\}$ . We show  $\min_{<} B_\xi \simeq \min_{<} (M_\xi[A_\gamma] \setminus A_\gamma)$ . Assume  $c := \min_{<} B_\xi \downarrow$ . If  $d \in k(c) \cap T_\xi$ , then  $k(d) \cap T_{<\xi} \subseteq k(c) \cap T_{<\xi} \subseteq A_\gamma$ , by minimality and  $d < c$  therefore  $d \in A_\gamma$ . It follows  $c \in M_\xi[A_\gamma]$ , and by  $M_\xi[A_\gamma] \subseteq B_\xi$ ,  $c = \min(M_\xi[A_\gamma] \setminus A_\gamma)$ . On the other hand, if  $c = \min(M_\xi[A_\gamma] \setminus A_\gamma)$ ,  $d < c$ ,  $d \in B_\xi$ , follows by induction on the length of  $e \forall e \in k(d) \cap T_\xi. e \in A_\gamma$  (using  $f \in k(e) \cap T_\xi \Rightarrow k(f) \cap T_{<\xi} \subseteq k(e)$ ), therefore  $d \in M_\xi[A_\gamma] \setminus A_\gamma$ , a contradiction, again  $c = \min B_\xi$ . Now it follows  $i_\gamma = i'_\gamma$ ,  $a_\gamma = a'_\gamma$ .

(b) Induction on  $\xi$ . By (a) it follows  $A_\gamma \cap T_\xi \subseteq M_\xi$ , by Remark 2.5 (c)  $A_\gamma \cap T_\xi$  is well-ordered, and by completeness and (a) it follows  $A_\gamma = \text{Acc}_\xi$ .

**Lemma 2.7.** *Let  $\tau_\xi$  be the  $\xi$ -th admissible ordinal. If  $\alpha \geq \tau_\xi$ , then  $A_\alpha$  is  $T_\xi$ -complete. Especially, the union of all  $A_\alpha$  is complete.*

*Proof.* Induction on  $\xi$ . By Remark 2.5 (e) it suffices to show the assertion for  $\kappa = \tau_\xi$ . By IH  $A_\kappa$  is  $T_\mu$ -complete ( $\mu < \xi$ ), therefore, if  $i_\kappa$  is defined, then  $\xi \leq i_\kappa$ . Assume  $c \in M_\xi[A_\kappa]$ ,  $M_\xi[A_\kappa] \cap c \subseteq A_\kappa$ . Then by the finiteness of  $k(c)$  there exists  $\alpha < \kappa$  such that  $c \in M_\xi[A_\alpha]$  and by  $\Sigma$ -collection relativized to  $L_\kappa$  it follows  $\forall \alpha < \kappa. \exists \beta < \kappa (M_\xi[A_\alpha] \cap c \subseteq A_\beta)$ . Further  $\forall \alpha < \kappa \exists \beta < \kappa. \alpha < \beta$  and  $\exists \alpha < \kappa. \alpha = 0$ . Application of Lemma 2.3 to the last three formulas,  $\exists \alpha < \kappa. c \in M_\xi[A_\alpha]$  and  $\forall \xi' < \xi. \tau_{\xi'} < \kappa$  yields that there exists  $\delta < \kappa$  such that the above conditions follow relativized to  $\delta$ , i.e.  $\delta$  is a limit ordinal such that  $\exists \alpha < \delta. c \in M_\xi[A_\alpha]$ ,  $\forall \alpha < \delta. \exists \beta < \delta (M_\xi[A_\alpha] \cap c \subseteq A_\beta)$  and  $\forall \xi' < \xi. \tau_{\xi'} < \delta$ . It follows  $\xi \leq i_\delta$ ,  $c \in M_\xi[A_\delta]$  and  $(M_\xi[A_\delta] \cap c \subseteq A_\delta)$ , therefore  $c \in A_\delta \subseteq A_\kappa$  or  $c \notin A_\delta$ ,  $i_\delta = \xi$ ,  $c = a_\delta \in A_\kappa$ .

Now we can substitute in the proof of Theorem 4.3 (a) of [18] with  $A := \bigcup_{\alpha \in \text{Ord}} A_\alpha \text{ Acc}_\mu$  by  $A \cap T_\mu$  and  $M_\mu$  by  $M_\mu[A]$  and replace in the equation which verifies that  $C^\nu(s)$  is definable in  $\text{ID}_\sigma$  “ $\forall \xi < \mu (k_{\nu, \xi}(r) \subseteq \text{Acc}_\xi)$ ” by  $\forall \xi \leq \mu (k_{\nu, \xi}(r) \subseteq \text{Acc}_\xi)$ ”, and show that  $\sigma$ -OS are well-ordered, provable in our variant of Kripke-Platek set theory.

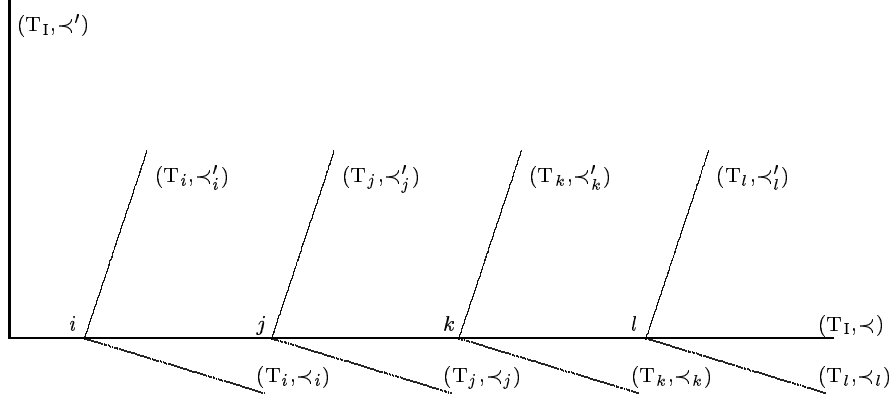
### 3 Ordinal Systems for one inaccessible

We are going to develop now ordinal systems for one inaccessible, in short I-OS, which will have in the limit order type  $|\text{KPI}| = |(\Delta_2^1 - \text{CA}) + (\text{BI})| = \psi(\epsilon_{1+1})$ . The step from  $\sigma$ -OS to I-OS is easy: instead of having one fixed order  $\Sigma$ , this order will now be generated by a further ordinal system.

Therefore we have OS-structures  $(T_a, <_a, <'_a, k_{a,a}, \text{length}_a)$  depending on  $a \in T_I$ , where  $T_I$  is the underlying set of terms of an OS-structure  $(T_I, <_I, <'_I, k_{I,I}, \text{length}_I)$ , functions  $k_{i,j} : T_i \rightarrow_\omega T_j$ , for  $i, j \in \mathcal{J} := T_I \cup \{I\}$ , an ordered set  $(L, <)$  together with functions  $\text{level}_i : T_i \rightarrow L$  and  $\tilde{k}_i : L \rightarrow_\omega T_i$  for  $i \in \mathcal{J}$ . As we did with  $\sigma$ -OS above we will simplify the definitions by replacing the individual component functions  $k_{i,j}$  by only one function  $k : T \rightarrow_\omega T$ , where  $T := \bigcup_{i \in \mathcal{J}} T_i$ , such that for  $a \in T_i$ ,  $k_{i,j}(a) = k(a) \cap T_j$ ; similarly we define only one function

$\tilde{k} : L \rightarrow_\omega T$ ,  $\text{level} : T \rightarrow L$  and  $\text{length} : T \rightarrow \omega$ , each. Further we will introduce global orders  $\prec, \prec'$  on  $T$ .

The resulting structure can now be visualized as follows:



The conditions are now almost the same as for  $\sigma$ -OS, except for the following two changes:

- If  $a \in T_i$ ,  $i \in T_I$ , then  $k(i) \subseteq k(a)$ . This guarantees that, if from ordinals generated before we can denote  $a$ , then we can already denote  $i$ . However, we can weaken this condition and will demand it only in case  $k_i(a) \neq \emptyset$  (since otherwise the information about  $i$  is already contained in the components of  $a$ ).
- $k_j(i) = \emptyset$  for  $i \prec j \in T_I$ . This guarantees that in the well-ordering proof using sequences, the sequence  $i_\gamma$  which corresponds to the sequence of  $i_\gamma$  in  $\sigma$ -ordinal systems is increasing, i.e. the elements of  $T_I$  are selected in increasing order.

**Definition 3.1.** (a) An *Ordinal-System-structure of type I* in short *I-OS-structure* is a 10-tuple

$$(T_I, (T_i)_{i \in T_I}, (\prec_i)_{i \in T_I \cup \{I\}}, (\prec'_i)_{i \in T_I \cup \{I\}}, \text{length}, k, L, \prec_L, \text{level}, \tilde{k})$$

such that, with

- $\mathcal{J} := T_I \cup \{I\}$ ,
- $T := \bigcup_{i \in \mathcal{J}} T_i$ ,

we have

- $T_I$  and  $T_i$  ( $i \in T_I$ ) are pairwise disjoint sets,
- $\prec_i, \prec'_i$  are linear orders on  $T_i$  for  $i \in \mathcal{J}$ ,
- $k : T \rightarrow_\omega T$ ,
- $\text{length} : T \rightarrow \mathbb{N}$ ,
- $(L, \prec_L)$  is a linearly ordered set,
- $\text{level} : T \rightarrow L$ ,
- $\tilde{k} : L \rightarrow_\omega T$ .

(b) For an I-OS-structure as above we define

- $k_i(a) := k(a) \cap T_i$ ;  $\tilde{k}_i(l) := \tilde{k}(l) \cap T_i$  ( $i \in \mathcal{J}$ ,  $a \in T$ ,  $l \in L$ );
- a binary relation  $\tilde{\prec}$  on  $\mathcal{J}$  defined for  $i, j \in \mathcal{J}$  by

$$i \tilde{\prec} j : \Leftrightarrow ((i \in T_I \wedge j \in T_I \wedge i \prec_I j) \vee (i \in T_I \wedge j = I)) ,$$

- a global order  $\prec$  on  $T$  defined for  $i, j \in \mathcal{J}$ ,  $a \in T_i$ ,  $b \in T_j$  by

$$\begin{aligned} a \prec b \Leftrightarrow & ((i = j \wedge a \prec_i b) \vee \\ & (i \in T_I \wedge j \in T_I \wedge i \prec_I j) \vee \\ & (i \in T_I \wedge j = I \wedge i \preceq_I b) \vee \\ & (i = I \wedge j \in T_I \wedge a \prec_I j)). \end{aligned}$$

- $a \prec' b : \Leftrightarrow a \prec_i b$  for  $a, b \in T_i$ .

- If  $A \subseteq T$ , then

$$k^{-1}(A) := \{a \in T \mid k(a) \subseteq A\},$$

$$\tilde{k}^{-1}(A) := \{a \in L \mid \tilde{k}(a) \subseteq A\}.$$

- If  $A \subseteq T$ ,  $a \in T$ , then  $A \cap a := \{b \in A \mid b \prec a\}$ .

(Note that as for  $\sigma$ -OS  $\prec$  should be considered only as a technical simplification: the real intuition should be that several OS-structures are working in parallel; further note that on  $T_i$  ( $i \in \mathcal{J}$ )  $\prec$  coincides with  $\prec_i$ ).

- (c) An I-OS-structure as above is an *I-times iterated Ordinal System* in short I-OS, if for all  $i, j, l \in \mathcal{J}$ ,  $i', j' \in T_I$ ,  $r, s \in T_i$ ,  $r', s' \in T_{i'}$  the following holds:

(I – OS 1)  $\text{length}[k(r)] < \text{length}(r)$ ;

(I – OS 2)  $k_i(r) \prec r$ ;

(I – OS 3) If  $r \prec s$ , then  $r \preceq k_i(s) \vee r \prec' s \vee \text{level}(r) \prec \text{level}(s)$ .

(I – OS 4) If  $j \tilde{\prec} i$ , then  $k_j \circ k_i \subseteq k_j$ .

(I – OS 5)  $\tilde{k}(\text{level}(r)) \subseteq k(r) \cup k[\text{level}[k_i(r)]]$ .

(I – OS 6) If  $i' \prec j'$ , then  $\text{level}[k_{j'}(r')] \prec \text{level}(r')$ .

(I – OS 7) If  $r \prec s$ , then  $\text{level}(r) \preceq_L \text{level}(s)$ .

(I – OS 8) If  $k_{i'}(r') = \emptyset$ , then  $k(i') \subseteq k(r')$ .

(I – OS 9) If  $i' \preceq j'$ , then  $k_{j'}(i') = \emptyset$ .

(I – OS 10) If  $A \subseteq T$  is  $\prec$ -well-ordered, then  $k^{-1}(A)$  is  $\prec'$ -well-ordered and  $\tilde{k}^{-1}(A)$  is  $\prec_L$ -well-ordered.

- (d) An I-OS as above is *well-ordered* if  $(T, \prec)$  is well-ordered, and its *order type* is that of  $(T_a, \prec_a)$ , where  $a$  is the  $\prec$ -minimal element of  $T_I$  such that  $T_a \neq \emptyset$ , (if no such  $a$  exists, the order type is 0).

It is *primitive recursive* if the involved sets are primitive recursive subsets of the natural numbers (parameterized in  $T_I$ , i. e.  $t \in T_u$  is primitive recursive in  $t$  and  $u$ ) all functions, relations (including  $\prec_I$ ) are primitive recursive and all properties (including linearity of  $\prec_I$ ,  $\prec'_I$ ) except for (I – OS 10) can be shown in primitive recursive arithmetic.

It is *elementary* if additionally the well-ordering condition follows in PRA in the sense of PRA-reducibility.

**Remark 3.2.** (I – OS 3), (I – OS 5) and (I – OS 8) could be replaced by the following conditions, which will be fulfilled in all the examples given in this article. We have chosen the weaker conditions above in order to be more general.

- (I – OS 3') If  $r \prec s$ , then  $r \preceq k_i(s) \vee r \prec' s$ .  
(I – OS 5')  $\tilde{k}(\text{level}(r)) \subseteq k(r)$ .  
(I – OS 8')  $k(i') \subseteq k(r')$ .

**Remark 3.3.** In an I-OS as above we have  $\forall a \in T_1.k(a) \prec a$ .

## 4 Well-ordering proofs for I-OS using sequences

We are going to show Theorem 6.9 (a), (b): I-OS are well-ordered, and, if they are elementary, they have order type below the proof theoretic strength of KPI. We will show our assertion by giving a well-ordering proof for I-OS, which for elementary I-OS can be formalized in KPI. As mentioned in the introduction, we will carry out these proofs in two versions: In this section we will carry out a well-ordering proof using sequences. We regard this proof as more intuitive than the proof using distinguished sets, which is suitable for constructive theories and carried out in Section 5.

The proof of the parts (a) and (b) of Theorem 6.9 will be almost identical. The latter one is achieved by carrying out the proof of (a) for elementary I-OS in KPI. The only difference between both proofs will be by choosing part (a) or (b) of the following assumption:

**Assumption 4.1.** In this and the next section assume an I-OS

$$\mathcal{O} = (T_1, (T_i)_{i \in \mathcal{J}}, (\prec_i)_{i \in \mathcal{J}}, (\prec'_i)_{i \in \mathcal{J}}, \text{length}, k, L, \prec_L, \text{level}, \tilde{k})$$

with the usual notations as above. Further assume one of the following:

- (a) Our meta-theory is at least Kripke-Platek set theory with natural numbers and there exists an admissible ordinal  $\kappa_0$  such that  $\forall \alpha < \kappa_0 \exists \pi < \kappa_0 (\alpha < \pi \wedge \pi$  is an admissible ordinal), where in the following admissible ordinal means “admissible ordinal  $> \omega$ ”.  $\mathcal{O} \in L_{\kappa_0}$  and all conditions can be verified in our meta-theory, where condition (I – OS 10) should be provable for all sets  $A \in L_{\kappa}$ . Let in this case  $\text{Ord} := \kappa_0$  and  $\text{Set} := L_{\kappa_0}$ , and sets and ordinals range over  $\text{Ord}$  and  $\text{Set}$ . This condition is fulfilled if  $\mathcal{O}$  is a set and we have as meta-theory ZFC.
- (b) Our meta-theory is at least Kripke Platek set theory with natural numbers,  $\forall \alpha \exists \kappa. \alpha < \kappa \wedge \kappa$  is an admissible ordinal,  $\mathcal{O}$  is a set, all conditions of being an I-OS can be verified in our meta-theory, where condition (I – OS 10) should be provable for all classes  $A$ . Let in this case  $\text{Ord} := \{\alpha \mid \alpha \text{ ordinal}\}$  and  $\text{Set} := \{x \mid x = x\}$ . This condition holds if we have KPI as meta-theory and  $\mathcal{O}$  is elementary.

The well-ordering proof will be similar to the one given in Section 2. Instead of selecting the elements of  $\Sigma$  in increasing order, we will now select for each  $\alpha$  the least element of  $T_1$  which has smaller components in the notations already defined before, but was not selected itself. For each such  $i$  we first select all elements of  $T_i$  as for  $\sigma$ -OS. Once we cannot find any more such an element, we select first  $i$  itself and then move on to the next  $i$ .

**Definition 4.2.** (a)  $M[A] := \{a \in T \mid k(a) \cap a \subseteq A\}$ .

For  $i \in \mathcal{J}$   $M_i[A] := M[A] \cap T_i$ .

(b) By recursion on the ordinals we define simultaneously  $i_\alpha \in T_I$  and  $a_\alpha \in T_{i_\alpha} \cup \{i_\alpha\}$ . (Note Convention 2.2.)

$$\begin{aligned} A_\alpha &:= \{a_\beta \mid \beta < \alpha \wedge a_\beta \downarrow\} \\ i_\alpha &:= \min_{\prec} (M_I[A_\alpha] \setminus A_\alpha) \\ a_\alpha &:= \begin{cases} \min_{\prec} (M_{i_\alpha}[A_\alpha] \setminus A_\alpha) & \text{if such a minimum exists,} \\ i_\alpha & \text{otherwise.} \end{cases} \end{aligned}$$

(c)  $\mathcal{W} := \{a_\alpha \mid \alpha \in \text{Ord} \wedge a_\alpha \downarrow\}$ .

$\mathcal{W}_i := \mathcal{W} \cap T_i$  ( $i \in \mathcal{J}$ ).

$\mathcal{W}_I^+ := \mathcal{W}_I \cup \{I\}$ .

**Remark 4.3.** If  $A \subseteq M[A]$ ,  $a \in M_i[A]$ ,  $i \in T_I$ , then  $i \in M[A]$ .

*Proof.* Induction on  $\text{length}(a)$ . If  $k_i(a) = \emptyset$ , then  $k(i) \subseteq k(a) \cap a$  and therefore the assertion follows. Otherwise  $c \in k_i(a)$  for some  $c$ ,  $c \in A \subseteq M[A]$ , by IH  $i \in M[A]$ .

**Lemma 4.4.** (a) If  $\alpha < \beta$ ,  $i_\beta \downarrow$ , then  $i_\alpha \preceq i_\beta$ .

(b) If  $\alpha < \beta$ ,  $a_\beta \downarrow$ , then  $a_\alpha \prec a_\beta$ .

(c)  $A_\alpha$  and  $\mathcal{W}$  are  $\prec$ -well-ordered, i.e. we have the principle of transfinite induction over these sets and classes.

*Proof.* (a): Ind on  $\beta$ . Assume  $\beta$  according to the assumption. If  $i_\beta \prec i_\alpha$ , then  $i_\beta \in M[A_\beta] \setminus M[A_\alpha]$ , there exists  $c \in k(i_\beta) \cap (A_\beta \setminus A_\alpha)$ . Let  $c = a_\gamma$  with  $\alpha \leq \gamma < \beta$ . By IH  $i_\beta \prec i_\alpha \preceq i_\gamma$ ,  $c = i_\gamma$  or  $c \in T_{i_\gamma}$  contradicting (I – OS 2) and (I – OS 9).

(b): If  $i_\alpha \prec i_\beta$ , this is clear. Otherwise  $i_\alpha = i_\beta$ ,  $i_\gamma = i_\alpha$  for  $\beta \leq \gamma \leq \alpha$  and the assertion follows similarly as in (a).

(c) by (b).

**Definition 4.5.** (a)  $i \in T_I$  can be *reached* by  $A$  iff  $i \in M_I[A]$  and  $M_I[A] \cap i \subseteq A$ .

(b) If  $i \in T_I$ , then  $a \in T_i$  can be *reached* by  $A$  iff  $i$  can be reached by  $A$ ,  $a \in M_i[A]$  and  $M_i[A] \cap a \subseteq A$  (equivalently: iff  $(M_i[A] \cup M_I[A]) \cap a \subseteq A$  and  $a, i \in M_i[A]$ ).

(c)  $A \subseteq T$  is  $T_I$ -complete ( $T_I \cap i$ -complete) iff, whenever  $i \in T_I$  ( $i \in T_I \cap i$ ) can be reached by  $A$ , then  $i \in A$ .

(d) Assume  $i \in T_I$ .  $A \subseteq T$  is  $T_i$ -complete iff it is  $T_I \cap i$ -complete and whenever  $a \in T_i$  can be reached by  $A$  then  $a \in A$ .

(e)  $A$  is *complete* iff it is  $T_I$ -complete and  $T_i$ -complete for every  $i \in T_I$ .

**Lemma 4.6.** (a)  $A_\alpha$  is  $T_I$ -complete iff  $i_\alpha \uparrow$  (and therefore as well  $a_\alpha \uparrow$ ).

(b) If  $i \in T_I$ , then  $A_\alpha$  is  $T_I \cap i$ -complete iff  $i_\alpha \uparrow \forall i \preceq i_\alpha$ .

(c) If  $i \in T_I$ , then  $A_\alpha$  is  $T_i$ -complete iff  $i_\alpha \uparrow \forall i \prec i_\alpha \vee a_\alpha = i$ .

(d) If  $A_\alpha$  is  $T_I$ -complete ( $T_I \cap i$ -complete,  $T_i$ -complete, complete) and  $\alpha < \beta$ , then  $A_\beta$  is  $T_I \cap i$ -complete ( $T_I \cap i$ -complete,  $T_i$ -complete, complete).

- (e) If  $\kappa$  is an admissible ordinal,  $\mathcal{O} \in L_\kappa$ ,  $i_\kappa \downarrow$ , then  $A_\kappa$  is  $T_{i_\kappa}$ -complete and therefore  $a_\kappa = i_\kappa$ .  
(f) If  $i \in T_I$ ,  $a \in \mathcal{W}_i$ , then  $k(i) \subseteq \mathcal{W}$ .  
(g)  $\mathcal{W}$  is complete.

*Proof.* (a), (b), (c): easy. (d): by (a) - (c) and Lemma 4.4 (a).

(e): Let  $i := i_\kappa$ . By (b)  $A_\kappa$  is  $T_I \cap i$ -complete. Assume  $b \in T_i$ ,  $b$  can be reached by  $A_\kappa$ . Then

$$\begin{aligned} i &\in M_I[A_\kappa] \wedge M_I[A_\kappa] \cap i \subseteq A_\kappa, \\ b &\in M_i[A_\kappa] \wedge M_i[A_\kappa] \cap b \subseteq A_\kappa. \end{aligned}$$

Similarly as in Lemma 2.7 it follows that for some  $\alpha < \kappa$

$$\begin{aligned} i &\in M_I[A_\alpha] \wedge M_I[A_\alpha] \cap i \subseteq A_\alpha, \\ b &\in M_i[A_\alpha] \wedge M_i[A_\alpha] \cap b \subseteq A_\alpha, \end{aligned}$$

$i_\alpha \preceq i$  therefore  $i_\alpha = i$ ,  $b \in A_\alpha \subseteq A_\kappa$  or  $b = a_\alpha \in A_\kappa$ .

(f) Induction on  $\text{length}(a)$ . If  $k_i(a) = \emptyset$ , then  $k(i) \subseteq k(a) \cap a \subseteq \mathcal{W}$ . Otherwise there exists  $c \in k_i(a) \subseteq \mathcal{W}$ , and by IH follows the assertion.

(g) Assume  $i \in M_I[\mathcal{W}] \wedge M_I[\mathcal{W}] \cap i \subseteq \mathcal{W}$ . As in (e) there exists some  $\alpha$  such that

$$i \in M_I[A_\alpha] \wedge M_I[A_\alpha] \cap i \subseteq A_\alpha.$$

It follows  $A_\alpha$  is  $T_I \cap i$ -complete. There exists  $\alpha < \kappa \in \text{Ord}$ ,  $\kappa$  an admissible ordinal. By (d)  $A_\kappa$  is  $T_I \cap i_\kappa$ -complete,  $i_\kappa \uparrow \forall i \preceq i_\kappa$ . If  $i_\kappa \uparrow \forall i \prec i_\kappa$ , it follows  $i \in A_\kappa \subseteq \mathcal{W}$ , otherwise  $i_\kappa = i$  and by (e)  $i = a_\kappa \in \mathcal{W}$ . If  $i$  can be reached by  $\mathcal{W}$ , then by the argument just given  $i = a_\alpha$  for some  $\alpha$ ,  $A_\alpha$  and therefore as well  $\mathcal{W}$  are  $T_i$ -complete.

**Lemma 4.7.** (a) If  $i \in T_I$  and  $a \in \mathcal{W}_i$ , then  $i \in M[\mathcal{W}]$  and  $M[\mathcal{W}] \cap i \subseteq \mathcal{W}$ .

(b) If  $a \in \mathcal{W}_i$ , then  $a \in M[\mathcal{W}]$  and  $M_i[\mathcal{W}] \cap a \subseteq \mathcal{W}$ .

*Proof.* (a)  $a = a_\gamma$  for some  $\gamma$ .  $i = i_\gamma = \min_{\prec} (M_I[A_\gamma] \setminus A_\gamma)$ ,  $i \in M_I[A_\gamma] \subseteq M_I[\mathcal{W}]$ , and by Lemma 4.4 (a), (I-OS 2) and (I-OS 9) it follows  $M_I[\mathcal{W}] \cap i = M_I[A_\gamma] \cap i \subseteq A_\gamma \subseteq \mathcal{W}$ . (b) follows in a similar way.

**Definition 4.8.** (a)  $\widetilde{\mathcal{W}} := k^{-1}(\mathcal{W})$ .

(b)  $\widetilde{L} := \widetilde{k}^{-1}(\mathcal{W})$ .

(c) For  $M \subseteq T$ ,  $M^{\preceq l} := \{a \in M \mid a \in T_I \vee \text{level}(a) \preceq l\}$ .

**Lemma 4.9.**  $\widetilde{\mathcal{W}} \cap T_I \subseteq \mathcal{W}$ .

*Proof.* Note  $\widetilde{\mathcal{W}} = M[\mathcal{W}]$ . We show  $\forall a \in \widetilde{\mathcal{W}}. a \in \mathcal{W}$  by main induction on  $\text{level}(a) \in \widetilde{k}^{-1}(\mathcal{W})$  (with respect to the well-ordering (!)  $\prec_L$  on it), side-induction on  $a \in \widetilde{\mathcal{W}}$  (with respect to the well-ordering (!)  $\prec'$  on it) and assume  $a$  according to the assertion.  $a \in M[\mathcal{W}]$ . We show  $\forall c \in M_I[\mathcal{W}] \cap a. c \in \mathcal{W}$  and assume  $c \in M_I[\mathcal{W}] \cap a$ . If  $\text{level}(c) \prec \text{level}(a)$ , by main IH  $c \in \mathcal{W}$ . Otherwise, if  $c \prec' a$ , then  $\text{level}(c) = \text{level}(a)$ , by side-IH  $c \in \mathcal{W}$ . Otherwise  $c \preceq k(a) \subseteq \mathcal{W}$ ,  $c \in M_I[\mathcal{W}] \cap \max(k(a)) \subseteq \mathcal{W}$ . Now  $a$  can be reached by  $\mathcal{W}$  and by the completeness of  $\mathcal{W}$  it follows  $a \in \mathcal{W}$ .

**Proof of Theorem 6.9 (a), (b) using sequences.**

First note that  $(\widetilde{\mathcal{W}}, \prec)$  and  $(\widetilde{\mathbb{L}}, \prec_L)$  are well-ordered.

We show by induction on  $l \in \widetilde{\mathbb{L}}$

$$\forall l \in \widetilde{\mathbb{L}}. (\mathbb{M}[\mathcal{W}])^{\preceq l} \subseteq \mathcal{W}$$

and assume  $l$  according to induction.

We show

$$\forall a \in \widetilde{\mathcal{W}}^{\preceq l}. a \in \mathcal{W}$$

by (side-)induction on  $(\widetilde{\mathcal{W}}, \prec')$ :

Assume  $i \in \mathcal{J}$ ,  $a \in \mathbb{M}_i[\mathcal{W}]$  according to the induction. We define for  $j \in \mathcal{J}$ ,  $\mathbb{C}^j(a) \subseteq \mathbb{T}_j$  and  $\mathbb{C}(a) \subseteq \mathbb{T}$ :

$$\mathbb{C}^j(a) := \begin{cases} \mathcal{W}_j \cap a & \text{if } j \widetilde{\prec} i, \\ \mathbb{M}_j[\mathcal{W}] \cap a & \text{if } j = i, \\ \{b \in \mathbb{T}_j \mid (j \neq \mathbb{I} \rightarrow \text{level}(b) \prec_L \text{level}(a)) \wedge \\ \quad \forall l \widetilde{\prec} i. (\mathbb{k}_l(b) \subseteq \mathcal{W}_l \cap a) \wedge \\ \quad \mathbb{k}_i(b) \subseteq \mathbb{M}_i[\mathcal{W}] \cap a \wedge \\ \quad \forall l \in \mathcal{J} (i \widetilde{\prec} l \widetilde{\prec} j \rightarrow \\ \quad \text{level}[\mathbb{k}_l(b)] \prec_L \text{level}(a))\} & \text{if } i \widetilde{\prec} j. \end{cases}$$

$$\mathbb{C}(a) := \bigcup_{j \in \mathcal{J}} \mathbb{C}^j(a).$$

Note that for  $i \widetilde{\prec} j$ ,

$$\mathbb{C}^j(a) = \{b \in \mathbb{T}_j \mid (j \neq \mathbb{I} \rightarrow \text{level}(b) \prec_L \text{level}(a)) \wedge \forall l \widetilde{\prec} j. \mathbb{k}_l(b) \subseteq \mathbb{C}^l(a)\} .$$

We prove

$$\forall j \in \mathcal{J}. i \widetilde{\prec} j \rightarrow \mathbb{k}[\mathbb{C}^j(a)] \subseteq \mathbb{C}(a) , \quad (1)$$

and assume  $i \widetilde{\prec} j$ ,  $b \in \mathbb{C}^j(a)$  and show  $\forall c \in \mathbb{k}(b). c \in \mathbb{C}(a)$  by induction on  $\text{length}(c)$ . Assume  $c \in \mathbb{T}_l$ . If  $l \widetilde{\prec} j$ , then by  $b \in \mathbb{C}^j(a)$  it follows  $c \in \mathbb{C}^l(a)$ . If  $l = j = i$ , then  $c \in \mathcal{W}_i \cap a \subseteq \mathbb{M}_i[\mathcal{W}] \cap a = \mathbb{C}(a)$ . Otherwise  $i \widetilde{\prec} j \widetilde{\prec} l \neq i$ . If  $l \neq \mathbb{I}$ ,  $\text{level}(c) \preceq_L \text{level}(b) \preceq_L \text{level}(a)$  and one of the  $\preceq_L$  in this chain is actually  $\prec_L$ , so  $\text{level}(c) \prec_L \text{level}(a)$ . In both cases  $l \neq \mathbb{I}$  and  $l = \mathbb{I}$  it follows for all  $k \widetilde{\prec} l$  that  $\mathbb{k}_k(c) \subseteq \mathbb{k}_k(b)$  and by IH  $\mathbb{k}_k(c) \subseteq \mathbb{C}(a)$ ,  $c \in \mathbb{C}^l(a)$ .

$$\forall j \in \mathcal{J}. i \widetilde{\prec} j \rightarrow \forall b \in \mathbb{C}^j(a). \widetilde{\mathbb{k}}(\text{level}(b)) \subseteq \mathbb{C}(a) \quad (2)$$

follows now by induction on  $\text{length}(b)$ .

$$\mathbb{M}_\mathbb{I}[\mathcal{W}] \cap a \subseteq \mathbb{C}(a) . \quad (3)$$

If  $a \in \mathbb{T}_\mathbb{I}$ , this follows by definition of  $\mathbb{C}^\mathbb{I}(a)$ . Assume  $a \in \mathbb{T}_i$ ,  $i \in \mathbb{T}_\mathbb{I}$ . We show  $\forall c \in \mathbb{M}_\mathbb{I}[\mathcal{W}] \cap a. c \in \mathbb{C}(a)$  by induction on  $\text{length}(c)$ . By Remark 3.3  $\mathbb{k}_\mathbb{I}(c) \subseteq$

$\mathcal{W} \subseteq M[\mathcal{W}]$  and by side-side-IH  $k_1(c) \subseteq C(a)$ . Further, if  $j \in T_1$ ,  $d \in k_j(c)$ , then  $j \prec c \prec i$ ,  $k_j(c) \subseteq \mathcal{W}_j \cap a = C^j(a)$ . It follows  $c \in C(a)$ .

We show by (side-side-)induction on  $\text{length}(b)$  simultaneously for all  $j$

$$\forall j \in \mathcal{J}. \forall b \in C^j(a). (b \in \mathcal{W} \wedge (i \not\prec_j b \rightarrow \text{level}(b) \in \tilde{L})) \quad (4)$$

and assume  $b$  according to induction,  $b \in C^j(a)$ . If  $j \prec i$ , this is trivial. Otherwise we have  $k(b) \subseteq C(a)$  and by IH  $k(b) \subseteq \mathcal{W}$ ,  $b \in \widetilde{\mathcal{W}}$ . Further  $\tilde{k}(\text{level}(b)) \subseteq k(b) \cup \tilde{k}[\text{level}[k_j(b)]] \subseteq \mathcal{W}$  by IH,  $\text{level}(b) \in \tilde{L}$ . Proof of  $b \in \mathcal{W}$ :

Case  $i = j$ :  $b \prec a$ , therefore  $b \preceq k_i(a)$  or  $\text{level}(b) \prec \text{level}(a)$  or  $b \prec' a$ . If  $b \preceq k_i(a)$ , by  $k_i(a) \subseteq \mathcal{W}$  and Lemma 4.7 (b) it follows  $b \in \mathcal{W}$ . If  $\text{level}(b) \prec_L \text{level}(a)$ , by main IH and  $\text{level}(b) \in \tilde{L}$  it follows  $b \in \mathcal{W}$ . Otherwise  $b \prec' a$ ,  $b \in \widetilde{\mathcal{W}}$  and by side-IH  $b \in \mathcal{W}$ .

Case  $i \prec j \neq 1$ :  $\text{level}(b) \prec \text{level}(a)$  and as before  $b \in \mathcal{W}$ .

Case  $i \prec j = 1$ :  $b \in \widetilde{\mathcal{W}}$ ,  $b \in \mathcal{W}$ .

By (3), (4),  $C^i(a) = M_i[\mathcal{W}] \cap a$ ,  $a \in M[\mathcal{W}]$ , therefore  $i \in T_1 \rightarrow i \in M[\mathcal{W}]$  and by the completeness of  $\mathcal{W}$  it follows  $a \in \mathcal{W}$ .

## 5 Well-ordering proofs using distinguished sets

In this section we are going to develop the technique for carrying out well-ordering proofs using distinguished sets. This is the technique used for analyzing constructive theories and subsystems of analysis without ordinals. In the case of standard ordinal notation systems of the strength of |KPI| it has been for instance used by [8] for a well-ordering proof in Feferman's theory  $T_0$  and in [17] for a well-ordering proof for Martin-Löf's Type Theory (which is slightly stronger than |KPI|). It was developed by Buchholz (the first presentation of it can be found in [1]).

A distinguished set is a set  $A$  which is built in the same way as the sets  $A_\gamma$  are defined. So  $A \cap T_1$  must be constructed by successively selecting the least element in  $M_1[A]$  not selected before, i.e.  $A$  is the set of elements of  $\text{Acc}(M_1[A])$  which are less than some  $a \in A$ . This can be formalized as  $A \cap T_1 = \text{Acc}(M_1[A])|A$ . ( $\text{Acc}(M)$  is the accessible part, i.e. largest well-ordered initial segment of the ordered set  $M$ ). Further,  $A \cap T_i$  is constructed in the same way and we get under the assumption that  $i$  can be reached, i.e.  $M_1[A] \cap i \subseteq A$ , the condition  $A \cap T_i = \text{Acc}(M_i[A])|A$ . We will need as well versions, where we relativize the set of ordinals needed in the definition of the accessible part and therefore get a  $\Sigma$ -definition of distinguished sets, which can be used in instances of  $\Sigma$ -collection.

**Definition 5.1.** (a) If  $A, B \subseteq T$ ,  $A|B := \{a \in A \mid \exists b \in B. a \preceq b\}$ .

(b) If  $A \subseteq T$ ,  $a \in T$ , then  $A|a := A|\{a\}$ .

(c)  $\text{Acc}_\alpha(A) := A_\alpha$ , where  $a_\gamma$  together with  $A_\gamma$  are defined by recursion on  $\gamma$  as  $a_\gamma := \min_{\prec}(A \setminus A_\gamma)$ ,  $A_\gamma := \{a_\beta \mid \beta < \gamma \wedge a_\beta \downarrow\}$ .

(d)  $\text{Acc}(A) := \text{Acc}_\kappa(A)$ , where  $\kappa$  is the least admissible such that  $A \in L_\kappa$ . One easily verifies that  $\text{Acc}(A)$  is the largest well-ordered segment of  $A$  i.e. the least set  $M$  such that  $a \in M$  iff  $\forall b \in A(b \prec a \rightarrow b \in M)$ .

- (e)  $W_I[A] := \text{Acc}(M_I[A])$ .  
If  $i \in T_I$ , then  $W_i[A] := \{a \in \text{Acc}(M_i[A]) \mid M_I[A] \cap i \subseteq A\}$ .  
 $W[A] := \bigcup_{i \in \mathcal{J}} W_i[A]$ .
- (f)  $W_I^\alpha[A] := \text{Acc}_\alpha(M_I[A])$ .  
If  $i \in T_I$ , then  $W_i^\alpha[A] := \{a \in \text{Acc}^\alpha(M_i[A]) \mid M_I[A] \cap i \subseteq A\}$ .  
 $W^\alpha[A] := \bigcup_{i \in \mathcal{J}} W_i^\alpha[A]$ .
- (g)  $A \subseteq T$  is *distinguished* (written as  $\text{Ag}(A)$ , where  $\text{Ag}$  stands for the original German word “ausgezeichnet” for “distinguished”) iff  $A = W[A]|A$ .
- (h)  $\text{Ag}_\alpha(A) := \Leftrightarrow \forall a \in (W_{\alpha+1}[A] \setminus W_\alpha[A])(A \prec a) \wedge A = W_\alpha[A]|A$ .
- (i) The class  $\mathcal{W}$  is the union of all distinguished sets which are elements of  $\text{Set}$ .  
 $\mathcal{W}^A$  is the union of all distinguished sets which are elements of  $A$ .

**Remark 5.2.** (a) If  $\forall a \in (W^{\alpha+1}[A] \setminus W^\alpha[A]).A \prec a$ , then  $W^\alpha[A]|A = W[A]|A$ .  
(b) If  $A$  is distinguished,  $A \in L_\kappa$ ,  $\kappa$  admissible, then there exists  $\alpha \prec \kappa$  such that  $\forall a \in (W^{\alpha+1}[A] \setminus W^\alpha[A]).A \prec a$ .  
(c) If  $A \in L_\kappa$ ,  $\kappa$  admissible, then  $A$  is distinguished iff  $\exists \alpha < \kappa. \text{Ag}_\alpha(A)$  iff  $\exists \alpha. \text{Ag}_\alpha(A)$ .  
(d) If  $B$  is an admissible set,  $a \in B$ ,  $\forall x \in a. \exists y \in B. (\text{Ag}(y) \wedge \phi(x, y))$ ,  $\phi(x, y)$  a  $\Sigma$ -formula, then there exists  $b \in B$  such that  $\forall x \in a. \exists y \in b. (\text{Ag}(y) \wedge \phi(x, y))$ .  
The same holds in case we have (b) of Assumption 4.1 with  $B$  replaced by the class of sets.

*Proof.* (a) It follows  $A \prec a$  for all  $a \in W^\beta[A] \setminus W^\alpha[A]$  and all  $\alpha \prec \beta$ .  
(b)  $\forall a \in A \exists \beta < \kappa. a \in W^\beta[A]$ . By  $\kappa$  admissible and  $A \in L_\kappa$  there exists  $\alpha < \kappa$  such that  $\forall a \in A. \exists \beta < \alpha. a \in W^\beta[A]$  and by  $A = W^\alpha[A]|A$ ,  $\forall a \in (W^{\alpha+1}[A] \setminus W^\alpha[A]).A \prec a$ .  
(c): By (a), (b). (d): Replace  $\text{Ag}(A)$  by the  $\Sigma$ -formula  $\exists \alpha. \text{Ag}_\alpha(A)$ .

**Lemma 5.3.** *If  $A$  is distinguished, then  $A$  is well-ordered.*

*Proof.* Let  $B := \{i \in T_I \mid i \in A \vee T_i \cap A \neq \emptyset\}$ .  
Assertion:  $\forall i \in B. B \cap i \subseteq A$ . Proof: Assume  $j \prec i \in B$ ,  $j \in B$  and show  $j \in A$ .  
If by the definition of  $j \in B$  we have  $j \in A$ , we are done. Otherwise there exists some  $c \in T_j \cap A$ .  $c \in M[A]$ ,  $j \in M[A]$ ,  $i \in W[A]$  or for some  $a \in T_i$ ,  $a \in W[A]$ , and with  $b = i$  or  $b = a$ ,  $j \in M_I[A] \cap b \subseteq A$ .  
 $B$  is well-ordered: Assume  $\forall y \in B. (\forall z \in B \cap y. \phi(z)) \rightarrow \phi(y)$ ,  $y \in B$ . Then  $B \cap y = A \cap T_I \cap y$ , which is well-ordered, therefore  $\forall z \in B \cap y. \phi(z)$ ,  $\phi(y)$ .  
Define now  $f : A \hookrightarrow \{\langle i, a \rangle \mid i \in B, a \in (A \cap T_i) \cup \{\infty\}\} =: C$ .  $T_I \ni a \mapsto \langle a, \infty \rangle$  and if  $i \in T_I$ ,  $T_i \ni a \mapsto \langle i, a \rangle$ . If we take as order on  $(A \cap T_i) \cup \{\infty\}$  the order  $(A \cap T_i, \prec) \oplus \{\infty\}$ , then  $C$  with the lexicographic order is well-ordered.  $f$  is order-preserving, therefore  $A$  is well-ordered.

**Lemma 5.4.** (a) Assume sets  $A, B$ ,  $\text{Ag}(A)$ ,  $\text{Ag}(B)$ ,  $a \in A$ ,  $a \preceq B$ . then  $a \in B$ .  
(b) If  $B$  is a set,  $\forall X \in B (X \subseteq T \wedge \text{Ag}(X))$ , then  $\text{Ag}(\bigcup B)$ .  
(c) A set  $A$  is distinguished iff  $A$  is an initial segment of  $\mathcal{W}$ , i.e.  $A = \mathcal{W}|A$ .  
(d)  $\mathcal{W}$  is well-ordered.

*Proof.* (a): We show  $\forall a \in A. (a \preceq B \rightarrow A|a = B|a)$  by induction on  $a$  and assume  $a$  according to induction. We show  $\forall b \in B. (b \prec a \rightarrow A|b = B|b)$  by induction on  $b$  and assume  $b$  according to induction. By main and side-IH it follows  $A \cap b = B \cap b$ ,  $b \in W[B]|b = W[B \cap b]|b = W[A \cap b]|b = W[A]|b = A|b$ . Therefore the side-induction is complete, together with main IH it follows  $A \cap a = B \cap a$  and the main induction step follows as the side induction step.

(b) If  $a \preceq \bigcup B$ ,  $a \preceq C$  for some  $C \in B$ , by (a)  $\bigcup B|a = C|a = W[C]|a = W[\bigcup B]|a$ , therefore  $\bigcup B = W[\bigcup B](\bigcup B)$ .

(c) If  $A$  is distinguished, then  $A \subseteq \mathcal{W}$  and if  $a \in \mathcal{W}|A$ ,  $a \in C$  for some distinguished set  $C$ , by (a)  $C|a = A|a$ ,  $a \in A$ . Therefore  $\mathcal{W}|A = A$ . If  $\mathcal{W}|A = A$ , then  $\forall x \in A. \exists C (\text{Ag}(C) \wedge x \in C)$ , By Remark 5.2 (d) there exists  $D$  such that  $\forall x \in A. \exists C \in D (\text{Ag}(C) \wedge x \in C)$ . Let  $D := \bigcup \{C|x \mid C \in D \wedge \text{Ag}(C) \wedge x \in C \cap A\}$ . Then  $D$  is distinguished,  $A \subseteq D$ ,  $D \subseteq A$  because  $A$  is an initial segment of  $\mathcal{W}$ ,  $A = D$ .

(d) Assume  $\forall x \in \mathcal{W}. (\forall y \in \mathcal{W} (y \prec x \rightarrow \phi(y)) \rightarrow \phi(x))$ ,  $a \in \mathcal{W}$ . Then  $a \in A$  for some distinguished set  $A$ ,  $A$  is an initial segment of  $\mathcal{W}$ , therefore  $\forall x \in A. (\forall y \in A (y \prec x \rightarrow \phi(y)) \rightarrow \phi(x))$ ,  $\forall x \in A. \phi(x)$ ,  $\phi(a)$ .

**Lemma 5.5.** (a)  $\mathcal{W}^A$  is distinguished.

(b) If  $A$  is distinguished,  $a \in T_i$ ,  $i \in T_I$ ,  $a$  can be reached by  $A$ , then  $(A \cap a) \cup \{a\}$  is distinguished.

(c) If  $A$  is distinguished,  $i \in T_I$ ,  $i$  can be reached by  $A$  and  $A$  is  $T_i$ -complete, then  $(A \cap i) \cup \{i\}$  is distinguished.

*Proof.* (a) By Lemma 5.4 (b).

(b) Let  $A' := (A \cap a) \cup \{a\}$ . We have  $W[A'] \cap a = A' \cap a$ : If  $b \in W[A'] \cap a$ , in case  $b \in T_i$ ,  $b \in M_i[A'] \cap a = M_i[A] \cap a \subseteq A \cap a \subseteq A'$ , in case  $b \in T_I$  again  $b \in A'$ , and in case  $b \in T_j$ ,  $j \neq i, I$ , then  $j \in M_I[A] \cap a \subseteq A \cap a$  and by  $A$  distinguished and  $b \in W[A] \cap j$ ,  $b \in A \cap a \subseteq A'$ . Further  $A' \cap a = A \cap a \subseteq W[A] \cap a = W[A'] \cap a$ .  $a \in W[A']$ ,  $a \in A'$ ,  $W[A']|A' = W[A']|a = A'$ .

(c) Let  $A' := (A \cap i) \cup \{i\}$ . As in (b) it follows  $W_I[A']|A' = A'$ ,  $W_j[A']|A' = A'$  for  $j \in T_I \setminus \{i\}$ . Further by the  $T_i$ -completeness of  $A$   $W_i[A'] = W_i[A] \subseteq A \cap i \subseteq A'$  therefore  $W_i[A']|A' = A'$ ,  $A'$  is distinguished.

**Lemma 5.6.** If  $A, B$  are distinguished sets,  $A \subseteq B$ ,  $a \in T$ ,  $a$  can be reached by  $A$ , then  $a$  can be reached by  $B$  as well.

*Proof.* Case  $a = i \in T_I$ : Let  $C := (T_I \cap i) \cup \bigcup_{j \in T_I \cap i} T_j$ . We show  $A \cap C = B \cap C$ : " $\subseteq$ " follows by  $A \subseteq B$ . " $\supseteq$ ": We show  $\forall b \in B \cap C. b \in A$  by induction on  $b$ . Subcase  $b \in T_I$ :  $b \in M[B]|b = M[B \cap b]|b = M[A \cap b]|b \subseteq M[A] \cap a \subseteq A$ . Subcase  $b \in T_j$ ,  $j \in T_I \cap C$ :  $b \in M[B]$ , therefore  $j \in M[B] \cap C$ , and as before  $j \in A$ ,  $b \prec j \in A$ ,  $b \in B$  and by Lemma 5.4 (a) it follows  $b \in A$ . Now  $i \in M[A] \subseteq M[B]$ , and  $M_I[B] \cap i = M_I[B \cap C] \cap i = M_I[A \cap C] \cap i \subseteq M[A] \cap i \subseteq A \subseteq B$ .

Case  $a \in T_i$ ,  $i \in T_I$ : We show  $A \cap a = B \cap a$ . " $\subseteq$ " is clear. " $\supseteq$ ": We show  $\forall b \in B \cap a. b \in A$  by induction on  $b$ . If  $b \in T_I$  or  $b \in T_j$ ,  $j \in T_j$ ,  $j \neq i$ , then the assertion follows by the case  $a \in T_I$  and since  $i$  can be reached by  $A$ . If  $b \in T_i$ , follows the assertion as in the case  $a \in T_I$ ,  $b \in T_I$ . As before follows now the assertion.

**Lemma 5.7.** (a) If  $A$  is an admissible set,  $i$  can be reached by  $\mathcal{W}^A$ , then  $\mathcal{W}^A$  is  $T_i$ -complete.  
(b)  $\mathcal{W}$  is complete.

*Proof.* (a) Assume  $b \in T_i$  can be reached by  $\mathcal{W}^A$ . Show  $b \in \mathcal{W}^A$ .  $i$  and  $b$  can be reached, i.e.

$$\begin{aligned} i &\in M_I[\mathcal{W}^A] \wedge M_I[\mathcal{W}^A] \cap i \subseteq \mathcal{W}^A \\ b &\in M_i[\mathcal{W}^A] \wedge M_i[\mathcal{W}^A] \cap b \subseteq \mathcal{W}^A \end{aligned}$$

By  $k(i) \cup k(b)$  finite, closure of distinguished sets under union and the assumption there exists  $B_0$  distinguished,  $B_0 \in A$  such that  $i, b \in M[B_0]$ . Further we can, using Remark 5.2 (d), inductively define distinguished sets  $B_{n+1} \in A$  ( $n \in \mathbb{N}$ ) such that  $B_n \subseteq B_{n+1}$ ,  $M_I[B_n] \cap i \subseteq B_{n+1}$ ,  $M_i[B_n] \cap b \subseteq B_{n+1}$ . Let  $B := \bigcup_{n \in \omega} B_n$ .  $B$  is distinguished,  $B \in A$ ,

$$\begin{aligned} i &\in M_I[B] \wedge M_I[B] \cap i \subseteq B \\ b &\in M_i[B] \wedge M_i[B] \cap b \subseteq B \end{aligned}$$

By Lemma 5.5 (b) it follows that  $(B \cap b) \cup \{b\}$  is distinguished and  $b \in \mathcal{W}_A$ .  
(b)  $\mathcal{W}$  is  $T_I$ -complete: Assume  $i$  can be reached by  $\mathcal{W}$ . As in (a) it follows that  $i$  can be reached by some distinguished set  $B$ . Let  $A$  be an admissible set such that  $B \in A$ . Then  $B \subseteq \mathcal{W}_A$ , by Lemma 5.6  $i$  can be reached by  $\mathcal{W}_A$ , by (a)  $\mathcal{W}_A$  is  $T_i$ -complete and by Lemma 5.5(c) it follows  $(\mathcal{W}_A \cap i) \cup \{i\}$  is distinguished,  $i \in \mathcal{W}$ . If  $i \in T_I$ ,  $i$  can be reached by  $\mathcal{W}$ , then as in (a) it follows that  $\mathcal{W}$  is  $T_i$ -complete, and the assertion follows.

**Lemma 5.8.** (a) If  $i \in T_I$  and  $a \in \mathcal{W}_i$ , then  $i \in M[\mathcal{W}]$  and  $M[\mathcal{W}] \cap i \subseteq \mathcal{W}$ .  
(b) If  $a \in \mathcal{W}_i$ , then  $a \in M[\mathcal{W}]$  and  $M_i[\mathcal{W}] \cap a \subseteq \mathcal{W}$ .

*Proof.* Trivial

**Proof of Theorem 6.9 (a), (b) using distinguished sets:** As before, using  $\mathcal{W}$  as defined in this section.

## 6 I-Ordinal Function Generators

We are going to show that the bound derived before is sharp, i.e. the supremum of the order types of elementary I-OS is  $|KPI|$ . As for the systems considered in [18] we will introduce I-ordinal function generators (I-OFGs), which provide a more general framework for defining I-OS. Using these we will define directly a sequence of OS, which could be regarded as “the standard I-OS”, in which we use essentially the  $n$ -times nested Schütte-Klammersymbole plus Cantor normal form, and which exhaust, although we will not prove that, the strength of  $|KPI|$ . We will then interpret directly the system used in [4] as an OFG and this will show the desired lower bound.

Note that the use of ordinals is not really necessary in order to develop the ordinal systems, it only simplifies the description of the ordinal systems slightly, and makes the relationship with the more traditional approach to ordinal notation systems more precise.

**Definition 6.1.** (a) An *I-ordinal function generator*, in short *I-OFG* is a quadruple

$$\mathcal{O} := (\mathcal{C}, \mathcal{L}, \mathcal{A}, \tilde{k}')$$

such that

- $\mathcal{C}$  is a set of ordinals,
- $\mathcal{L} = (L', \prec_{L'})$  is a well-ordering,
- $\mathcal{A} = (\text{Arg}_\kappa, \prec'_\kappa, k'_\kappa, l'_\kappa, \text{level}'_\kappa)_{\kappa \in \mathcal{C} \cup \{I\}}$  ( $I$  a special symbol),
- $\tilde{k}' : L' \rightarrow_\omega \text{Ord}$ ,
- for  $\kappa \in \mathcal{C} \cup \{I\}$  we have
  - $(\text{Arg}_\kappa, \prec'_\kappa)$  is a well-ordering,
  - $k'_\kappa, l'_\kappa : \text{Arg}_\kappa \rightarrow_\omega \text{Ord}$ ,
  - $\text{level}'_\kappa : \text{Arg}_\kappa \rightarrow L'$ ,
  - $\forall a \in \text{Arg}_\kappa. l'_\kappa(a) \subseteq k'_\kappa(a)$ ,
  - $\forall a, b \in \text{Arg}_\kappa. (a \prec'_\kappa b \rightarrow \text{level}'_\kappa(a) \preceq_{L'} \text{level}'_\kappa(b))$ ,
  - $\forall a \in \text{Arg}_\kappa. \tilde{k}'[l'_\kappa(a)] \subseteq k'_\kappa(a)$ ,
- $\text{Arg}_\kappa$  ( $\kappa \in \mathcal{C} \cup \{I\}$ ) are pairwise disjoint,

We assume in the following that  $L'$  and  $\text{Arg}_\kappa$  are disjoint and omit therefore the index  $\kappa$  in  $k', l', \prec', \text{level}'$  and the index  $L'$  in  $\prec_{L'}$ . Further let  $\kappa, \mu, \nu$  be, unless mentioned differently, elements of  $\mathcal{I} := \mathcal{C} \cup \{I\}$ , and let  $\mathcal{O}$  always be as above.

- (b) For  $\kappa \in \mathcal{C}$   $\kappa^- := \bigcup_{\pi \in \mathcal{C} \cap \kappa} \pi$  and  $O_\kappa := \kappa \setminus (\kappa^- + 1)$ .

$$I^- := 0, O_I := \mathcal{C}.$$

Define the relation  $\tilde{\sim}$  on  $\mathcal{I}$  by  $\kappa \tilde{\sim} \mu \Leftrightarrow (\kappa, \mu \in \mathcal{C} \wedge \kappa < \mu) \vee (\kappa \in \mathcal{C} \wedge \mu = I)$ . Let  $\Lambda$  be minimal such that  $\mathcal{C} < \Lambda$ . (In order to avoid the existence of large cardinals, one can assume that  $\mathcal{C}$  is a class instead of a set and treat  $\Lambda$  as a symbol for undefined).

$$\text{Arg} := \bigcup_{\kappa \in \mathcal{I}} \text{Arg}_\kappa.$$

- (c) An I-OFG is *cardinality based* iff

- $\mathcal{C}$  is a set of cardinals which has as order type a regular cardinal  $\pi$  such that  $\omega < \pi \wedge \forall \kappa \in \mathcal{C}. \kappa < \pi$  (if we have treated  $\Lambda$  as undefined above, we demand that  $\mathcal{C}$  is a proper class),
- for all  $\kappa$  such that  $\kappa$  is not a regular cardinal  $> \omega$  we have  $\text{Arg}_\kappa = \emptyset$ ,
- for all  $B \subseteq \text{Ord}$  countable  $k'^{-1}_\kappa(B)$  is countable,
- for  $a \in \text{Arg}_\kappa, l'_\kappa(a) \subseteq k'_\kappa(a) \cap O_\kappa$ .

In this case we define for  $a \in \text{Arg}_\kappa, k'_{\kappa, \nu}(a) := k'_\kappa(a) \cap O_\nu$ .

- (d) If  $\mathcal{O}$  as above is a cardinality based I-OFG, we define simultaneously for all  $\kappa \in \mathcal{I}$  for  $a \in \text{Arg}_\kappa$  by recursion on  $\text{level}'_\kappa(a) \in L'$ , side-recursion on  $(\text{Arg}_\kappa, \prec'_\kappa)$ ,  $\text{eval}_\kappa(a) \in \text{Ord}$ , subsets  $C_\kappa(a), C_\kappa^n(a) \subseteq \text{Ord}$  ( $n \in \omega$ ) and, whether  $\text{NF}_\kappa(a)$  holds, by:

Case  $\kappa \in \mathcal{C}$ :

- $C_\kappa^0(a) := (\kappa^- + 1) \cup (\bigcup (k'_\kappa(a) \cap O_\kappa)) \cup l'_\kappa(a)$ ,
- $C_\kappa^{l+1}(a) := C_\kappa^l(a) \cup \{ \text{eval}_\nu(b) \mid \nu = \text{IV}(\nu = \text{eval}_I(c) \in C_\kappa^l(a) \wedge \text{NF}(c) \wedge c \in \text{Arg}_I) \wedge \kappa \tilde{\sim} \nu \wedge b \in \text{Arg}_\nu \wedge ((\nu = \kappa \wedge b \prec'_\kappa a) \vee \text{level}'(b) \prec_{L'} \text{level}'(a)) \wedge k'(b) \subseteq C_\kappa^l(a) \wedge \text{NF}_\nu(b) \}$ ,

$$- \text{eval}_\kappa(a) := \min(\{\alpha \mid \alpha \notin C_\kappa(a)\} \cup \{\kappa\}).$$

Case  $\kappa = \text{I}$ :

$$\begin{aligned} - C_{\text{I}}^0(a) &:= (\bigcup (\mathbf{k}'_1(a) \cap \text{O}_1)) \cup \mathbf{l}'_1(a), \\ - C_{\text{I}}^{l+1}(a) &:= C_{\text{I}}^l(a) \cup \\ &\quad \bigcup \{\text{eval}_1(b) + 1 \mid b \in \text{Arg}_1 \wedge b \prec' a \wedge \\ &\quad \wedge \mathbf{k}'(b) \subseteq C_{\text{I}}^l(a) \wedge \text{NF}_1(b)\}, \\ - \text{eval}_1(a) &:= \min(\{\alpha \in \mathcal{C} \mid \alpha \notin C_\kappa(a)\} \cup \{A\}). \end{aligned}$$

In both cases

$$\begin{aligned} - C_\kappa(a) &:= \bigcup_{n \in \omega} C_\kappa^n(a), \\ - \text{NF}_\kappa(a) &:\Leftrightarrow \mathbf{k}'(a) \subseteq C_\kappa(a). \end{aligned}$$

- (e)  $\text{eval}(a) := \text{eval}_\kappa(a)$  for  $a \in \text{Arg}_\kappa$ ,  $\kappa \in \mathcal{I}$ .  
 $\text{NF}_\kappa := \{a \in \text{Arg}_\kappa \mid \text{NF}_\kappa(b)\}$ ,  $\text{NF} := \bigcup_{\kappa \in \mathcal{I}} \text{NF}_\kappa$ .  
For  $a, b \in \text{Arg}$   $a \prec b \Leftrightarrow \text{eval}(a) < \text{eval}(b)$ .

- (f) Let  $\mathcal{O}$  be a cardinality based I-OFG.

We define  $\text{T} \subseteq \text{NF}$  inductively by: If  $\kappa = \text{I} \vee (c \in \text{T} \cap \text{Arg}_1 \wedge \kappa = \text{eval}(c))$ ,  $b \in \text{NF}_\kappa$ ,  $\mathbf{k}'(b) \subseteq \text{eval}[\text{T}]$ , then  $b \in \text{T}$ .

$$\text{T}_1 := \text{T} \cap \text{Arg}_1. \quad \text{T}_a := \text{T} \cap \text{Arg}_{\text{eval}_1(a)} \text{ for } a \in \text{T}_1.$$

$$\text{Arg}[\text{T}] := \{a \in \text{Arg} \mid \mathbf{k}'(a) \subseteq \text{eval}[\text{T}]\}, \quad \text{Arg}_\kappa[\text{T}] := \text{Arg}[\text{T}] \cap \text{Arg}_\kappa.$$

Note  $\text{T} \subseteq \text{Arg}[\text{T}]$ .

Assuming that  $\text{eval} \upharpoonright \text{NF}$  is injective and  $n \subseteq \text{eval}(\text{T}_1)$ , which will be shown below, we define  $\mathbf{k}^0, \mathbf{k}^1, \mathbf{k} : \text{Arg}[\text{T}] \rightarrow_\omega \text{T}$ ,  $\text{length} : \text{Arg}[\text{T}] \rightarrow \mathbb{N}$ ,  $\text{level} : \text{Arg}[\text{T}] \rightarrow L'$  simultaneously by induction on  $a \in \text{Arg}[\text{T}]$  as follows: Assume  $a \in \text{Arg}_\kappa[\text{T}]$ .

$$\begin{aligned} - \kappa \in \mathcal{C} &\Rightarrow \mathbf{k}^0(a) := (\text{eval}^{-1}(\mathbf{k}'(a)) \cap \text{T}) \cup \mathbf{k}(\text{eval}^{-1}(\kappa)). \\ - \kappa = \text{I} &\Rightarrow \mathbf{k}^0(a) := \text{eval}^{-1}[\mathbf{k}'(a)] \cap \text{T}. \\ - \mathbf{k}^1(a) &:= \mathbf{k}^0(a) \cup \{d \in \mathbf{k}(c) \mid c \in \mathbf{k}^0(a) \wedge d \prec c\}. \\ - \mathbf{k}(a) &:= \mathbf{k}^1(a) \cup \mathbf{k}[\mathbf{k}^1(a) \cap \text{Arg}_\kappa]. \\ - \text{length}(a) &:= \max(\text{length}[\mathbf{k}'(a)] \cup \{-1\}) + 1. \\ - \text{level}(a) &:= \max(\{\text{level}'(a)\} \cup \text{level}[\mathbf{k}_{\kappa, \kappa}(a)]). \end{aligned}$$

- (g)  $\mathcal{J} := \text{T}_1 \cup \{\text{I}\}$ .

In the following let always  $k, l \in \mathcal{J}$ .

Let  $\prec_k, \prec'_k$  be the restriction of  $\prec, \prec'$  to  $\text{T}_k$ .

$$\mathbf{l}(a) := \text{eval}^{-1}(\mathbf{l}'(a)).$$

$$\mathbf{k}_k(a) := \mathbf{k}(a) \cap \text{T}_k, \quad \mathbf{l}_k(a) := \mathbf{l}(a) \cap \text{T}_k \quad (k \in \mathcal{J}).$$

$$C_{\text{I}}^l(a) := \text{eval}^{-1}(C_{\text{I}}(a)), \text{ and for } k \in \text{T}_1, \quad C_k^l(a) := \text{eval}^{-1}(C_{\text{eval}(k)}(a)).$$

$$\text{L} := \tilde{\mathbf{k}}^{-1}(\text{eval}[\text{T}]).$$

$\prec_L$  be the restriction of  $\prec_{L'}$  to  $\text{L}$ .

$$\text{Define } \tilde{\mathbf{k}} : \text{L} \rightarrow_\omega \text{T}, \quad \tilde{\mathbf{k}}(a) := \text{eval}^{-1}(\tilde{\mathbf{k}}'(a)) \cap \text{T}.$$

If  $\text{eval}(k)^-$  exists,  $k^- := \text{eval}^{-1}(\text{eval}(k)^-)$ .

**Lemma 6.2.** *Let  $\mathcal{O}$  be a cardinality based I-OFG.*

- (a)  $C_\kappa(a)$  is the least set  $M$  such that  $C_\kappa^0(a) \subseteq M$  and such that, if  $\nu = \text{I} \vee (\nu = \text{eval}_1(c) \in C_\kappa^l(a) \wedge \text{NF}(c) \wedge c \in \text{Arg}_1)$ ,  $\kappa \tilde{\prec} \nu$ ,  $b \in \text{Arg}_\nu$ ,  $\text{NF}_\nu(b)$ ,  $\mathbf{k}'(b) \subseteq M$ ,  $(\nu = \kappa \wedge b \prec' a) \vee \text{level}'(b) \prec \text{level}'(a)$ , then  $\text{eval}_\nu(b) \in M$ .

- (b) If  $\kappa \in \mathcal{C}$ ,  $a \in \text{Arg}_\kappa$ , then  $C_\kappa(a) \cap O_\kappa$  is an initial segment of  $O_\kappa$ .  
*Epecially*  $\text{eval}_\kappa(a) = C_\kappa(a) \cap \kappa$ .  
 If  $a \in \text{Arg}_I$ , then  $C_I(a)$  is an initial segment of  $\text{Ord}$ .
- (c) If  $a \in \text{Arg}_\kappa$ , then  $\text{eval}_\kappa(a) \in O_\kappa$  ( $\kappa \in \mathcal{I}$ ).
- (d) If  $a, b \in \text{NF}_\kappa$ , then  $\text{eval}(a) < \text{eval}(b) \Leftrightarrow (a \prec' b \wedge k'_{\kappa, \kappa}(a) < \text{eval}(b)) \vee \text{eval}(a) \leq k'_{\kappa, \kappa}(b)$ .  
*Epecially*  $\text{eval} \upharpoonright \text{NF}_\kappa$  is injective and therefore  $k, k^0, k^1$ , length and level are well-defined.
- (e)  $\mathcal{F} := (\mathbb{T}_I, (\mathbb{T}_k)_{k \in \mathbb{T}_I}, (\prec_k)_{k \in \mathcal{J}}, (\prec'_k)_{k \in \mathcal{J}}, \text{length}, k, L, \prec_L, \text{level}, \tilde{k})$  is an I-OS-structure.
- (f) In  $\mathcal{F}$ , the relation  $\prec$  derived from  $\mathcal{F}$  being an I-OS-structure coincides with the original relation  $\prec$ .

*Proof.* (a) follows, since  $k'(a)$  is finite.

(b): Main induction on  $\text{level}(a)$ ,  $\prec'$ -side induction on  $a$ . We show  $\forall \alpha \in C_\kappa^n(a) \cap \kappa. \alpha \subseteq C_\kappa(a)$  by side-side-induction on  $n$ :  $n = 0$ : clear, since  $l'(a) \subseteq k'_{\kappa, \kappa}(a)$ .  $n \rightarrow n + 1$ : Case  $\kappa = I$ : clear. Otherwise: If  $\alpha \in (C_\kappa^{n+1}(a) \setminus C_\kappa^n(a)) \cap O_\kappa$ , then  $\alpha = \text{eval}_\kappa(b)$  with  $k'(b) \subseteq C_\kappa^n(a)$  and  $b \prec' a$ , by side-side-IH and, since  $l'(b) \subseteq k'(b)$ ,  $\bigcup(k'(b) \cap O_\kappa) \cup l'(b) \subseteq C_\kappa^n(a)$ , and by an immediate induction  $C_\kappa^l(b) \subseteq C_\kappa(a)$  for all  $l \in \omega$ ,  $C_\kappa(b) \subseteq C_\kappa(a)$ , therefore  $\alpha = \min\{\gamma \mid \gamma \notin C_\kappa(b)\} \subseteq C_\kappa(a)$ .

(c)  $\kappa \neq I$ . By an induction and the regularity of  $\kappa$  it follows that the cardinality of  $C_\kappa^n(a)$  and therefore as well that of  $C_\kappa(a)$  is  $< \kappa$ . We conclude that there exists an  $\alpha < \kappa$  such that  $\alpha \notin C_\kappa(a)$ .

$\kappa = I$ : Similarly it follows that there the cardinality of  $C_I(a)$  is  $< \Lambda$ . Therefore there exists  $\alpha < \Lambda$  such that  $\alpha \notin C_I(a)$ . For some  $\alpha < \pi \in \mathcal{C}$  it follows  $\pi \notin C_I(a) \cap \mathcal{C}$ .

(d) “ $\Leftarrow$ ” If  $a \prec' b$ ,  $k'_{\kappa, \kappa}(a) < \text{eval}(b)$ ,  $\text{NF}(a)$ , it follows easily  $k'(a) \subseteq C_\kappa(a) \subseteq C_\kappa(b)$ ,  $\text{eval}(a) \in C_\kappa(b) \cap \kappa = \text{eval}(b)$ . If  $\text{eval}(a) \leq k'_{\kappa, \kappa}(b)$ , the assertion follows by  $k'_{\kappa, \kappa}(b) < \text{eval}(b)$ . “ $\Rightarrow$ ” If the right side is false, the right side holds for  $b \prec a$  or  $a = b$ , so the left side is false.

(e) Trivial. (f) Easy.

**Convention 6.3.** In the following we will use all the definitions introduced for general I-OS-structures for the I-OS-structure of Lemma 6.2 (e).

**Lemma 6.4.** Assume  $k \in \mathcal{J}$ ,  $a \in \mathbb{T}_k$ .

- (a)  $b \in k(a) \Rightarrow k(b) \cap b \subseteq k(a)$ .  
 (b)  $b \in k_k(a) \Rightarrow k(b) \subseteq k(a)$ .  
 (c)  $k = I \Rightarrow k(a) \prec a$ .  
 (d)  $(k \in \mathbb{T}_I \wedge b \in C'_k(a) \setminus (k^- \cup \{k^-\})) \Rightarrow k(b) \subseteq C'_k(a)$ .  
 (e)  $(k \in \mathcal{J} \wedge a \in \mathbb{T}_k) \Rightarrow k(a) \subseteq C'_k(a)$ .  
 (f) If  $l \in \mathbb{T}_I$ ,  $k \tilde{\prec} l$ ,  $b \in \mathbb{T}_l \cap C'_k(a)$ , then  $\text{level}(b) \prec \text{level}(a)$ .  
 (g)  $\text{length}[k(a)] < \text{length}(a)$ .  
 (h)  $k_k(a) \prec a$ .  
 (i)  $(b \in \mathbb{T}_k \wedge a \prec b) \Rightarrow (a \prec' b \vee a \preceq k_k(b))$ .

- (j) If  $k, l \in \mathcal{J}$ ,  $l \tilde{\prec} k$ , then  $k_l[k_k(a)] \subseteq k(a)$ .
- (k)  $k(\text{level}(a)) \subseteq k(a)$ .
- (l)  $k \prec l \in \mathbf{T}_I \Rightarrow \text{level}[k_l(a)] \prec \text{level}(a)$ .
- (m)  $(a, b \in \mathbf{T}_k \wedge a \prec b) \Rightarrow \text{level}(a) \preceq \text{level}(b)$ .
- (n)  $(k \in \mathbf{T}_I \wedge a \in \mathbf{T}_k) \Rightarrow k(k) \subseteq k(a)$ .
- (o)  $(k, l \in \mathbf{T}_I \wedge k \prec l) \Rightarrow k_l(k) = \emptyset$ .
- (p)  $\mathcal{F}$  is an I-OS. We call any I-OS isomorphic to  $\mathcal{F}$  an I-OS based on  $\mathcal{O}$ .

*Proof.* (a), (b): Induction on  $\text{length}(b)$ .

(c): Induction on  $\text{length}(a)$ .  $k^1(a) \preceq k^0(a) \subseteq C'_1(a)$ , therefore  $k^1(a) \subseteq C'_1(a)$ . By IH  $k[k^1(a) \cap \text{Arg}_\kappa] \prec k^1(a) \prec a$ .

(d) Induction on  $\text{length}(b)$ :  $k^0(b) \subseteq C'_k(a)$  and by IH it follows the assertion.

(e) In case  $k = I$  this follows by  $k(a) \prec a$  and otherwise it follows similarly as (d), but using (d) instead of the IH.

(f) Induction on  $\text{length}(b)$ . By (c), (d),  $k_l(b) \subseteq C_k(a)$  and by IH therefore  $\text{level}[k_l(b)] \prec \text{level}(a)$ . Further  $\text{level}'(b) \prec \text{level}'(a) \preceq \text{level}(a)$ , and therefore  $\text{level}(b) \prec \text{level}(a)$ .

(g) Trivial. (h) By  $k(a) \subseteq C'_k(a)$ . (i) Lemma 6.2 (d).

(j) If  $l \prec k \in \mathbf{T}_I$ ,  $b \in k_k(a)$ , then  $k_l(b) \subseteq k(b) \cap b \subseteq k(a)$  by (a). If  $l \in \mathbf{T}_I$ ,  $k = I$ ,  $b \in k_k(a)$ , then by (c)  $k(b) \prec b$ ,  $k(b) = k(b) \cap b \subseteq k(a)$ .

(k) Note first that for all  $a$   $\tilde{k}(\text{level}'(a)) \subseteq k(a)$ . We show by induction on  $\text{length}(a)$  the assertion:  $\tilde{k}(\text{level}(a)) \subseteq \tilde{k}(\text{level}'(a)) \cup k[\text{level}[k_k(a)]] \subseteq k(a) \cup k[k_k(a)] \subseteq k(a)$ .

(l): By (c), (e), (f).

(m) Induction on  $\text{length}(a) + \text{length}(b)$ . If  $a \preceq k_k(b)$ , then by IH  $\text{level}(a) \preceq \text{level}[k_k(b)] \preceq \text{level}(b)$ . Otherwise  $a \prec' b$ ,  $\text{level}'(a) \preceq \text{level}'(b) \preceq \text{level}(b)$ . Further by IH  $\text{level}[k_k(a)] \preceq \text{level}(b)$  and therefore  $\text{level}(a) \preceq \text{level}(b)$ .

(n) Trivial. (o) By (d). (p) By the above.

**Definition 6.5.** (a)  $\Omega_0 := 0$ ,  $\Omega_\alpha := \aleph_\alpha$  for  $\alpha \neq 0$ .

(b) Let  $\mathbf{I}$  be the least strongly inaccessible cardinal. (One can easily replace this by demanding that the class of ordinals has the properties of a strongly inaccessible cardinal).

(c) Define  $0_k \in \mathcal{S}_k$  by  $0_0 := 0$ ,  $0_{k+1} := \left( \right)$  (the empty Klammersymbol), where  $\mathcal{S}_k$  is as in Example 2.7. (i) of [18].

**Example 6.6.** (a) Let  $l \in \omega$ ,  $l > 0$  be fixed. We define an OFG as follows:

–  $\mathcal{C} := \{\Omega_\alpha \mid 0 < \alpha < \mathbf{I}\}$ , (where  $\Omega_\alpha := \aleph_\alpha$  for  $\alpha > 0$ ),

$(\text{Arg}_{\Omega_\alpha}, \prec'_{\Omega_\alpha}) := \emptyset$  if  $\alpha$  is not a successor ordinal,

$$(\text{Arg}'_{\Omega_{\alpha+1}}, \prec''_{\Omega_{\alpha+1}}) := \begin{cases} \text{CNF}_{\Omega_1}([0, \Omega_1[, ]0, \omega]), & \text{if } \alpha = 0, \\ \text{CNF}_{\Omega_{\alpha+1}}([0, \Omega_{\alpha+1}[, ]0, \Omega_\alpha]) \setminus \\ \quad (\text{CNF}_{\Omega_{\alpha+1}}(\{0\}, ]0, \Omega_\alpha]) \cup \\ \quad \{\text{CNF}_{\Omega_{\alpha+1}}(1, 1)\} & \text{otherwise,} \end{cases}$$

$(\text{Arg}_{\Omega_{\alpha+1}}, \prec'_{\Omega_{\alpha+1}}) := (\text{Arg}'_{\Omega_{\alpha+1}}, \prec''_{\Omega_{\alpha+1}}) \otimes \psi_{\Omega_{\alpha+1}}(\mathcal{S}_l)$ ,

where  $\mathcal{S}_l$  is defined as in Example 2.7. (i) of [18], but restricted to ordinals  $< \mathbf{I}$ .

- $(\text{Arg}'_{\mathbf{I}}, \prec'_{\mathbf{I}}) := \Omega_{[0, \mathbf{I}]} \otimes \psi_{\mathbf{I}}(\mathcal{S}_{\mathbf{I}})$ .
- $k'_{\Omega_{\alpha+1}}(a), l'_{\Omega_{\alpha+1}}$  are defined as  $k_i, l_i$  in Example 3.7. (b) of [18], for  $\alpha = 0$  as  $i = 0$  and for  $\alpha > 0$  as  $i > 0$ ,  
 $l'_{\mathbf{I}}(\Omega_{\mathbf{I}}) := k'_{\mathbf{I}}(\Omega_{\mathbf{I}}) := \emptyset$ ,  
for  $\alpha > 0$ ,  $k'_{\mathbf{I}}(\Omega_{\alpha}) := \{\alpha\}$ ,  $l'_{\mathbf{I}}(\Omega_{\alpha}) := \emptyset$ ,  
 $k'_{\mathbf{I}}(\psi_{\mathbf{I}}(A)) := k'_0(A)$ , where  $k'_0$  is as in Example 2.7. (i) of [18],  $l'_{\mathbf{I}}(\psi_{\mathbf{I}}(A)) := \emptyset$ .
- $(L', \prec_{L'}) := \mathcal{S}_{\mathbf{I}}$  (with its order).
- $\text{level}_{\Omega_{\alpha+1}}(\text{CNF}'_{\Omega_{\alpha+1}}(\vec{a})) := 0_l$ ,  
 $\text{level}_{\Omega_{\alpha+1}}(\psi_{\Omega_{\alpha+1}}(A)) := A$ ,  
 $\text{level}_{\mathbf{I}}(\Omega_{\alpha}) := 0_l$ ,  
 $\text{level}_{\mathbf{I}}(\psi_{\mathbf{I}}(A)) := A$ .
- $\tilde{k}(A) := k'_0(A)$ .

Then one easily sees that the above definitions yield a cardinality based I-OFG ( $\kappa_0 = \mathbf{I}$ ). It follows  $\text{eval}_{\mathbf{I}}(\Omega_{\alpha}) = \Omega_{\alpha}$  ( $\alpha > 0$ ),

$$\text{eval}_{\Omega_{\mathbf{I}}}(\text{CNF}_{\Omega_{\mathbf{I}}}(\alpha_1, m_1, \dots, \alpha_n, m_n)) = \omega^{\alpha_1} m_1 + \dots + \omega^{\alpha_n} m_n,$$

$$\text{eval}_{\Omega_{\alpha+1}}(\text{CNF}_{\Omega_{\alpha+1}}(\alpha_1, \beta_1, \dots, \alpha_n, \beta_n)) = \Omega_{\alpha}^{\alpha_1} \beta_1 + \dots + \Omega_{\alpha}^{\alpha_n} \beta_n.$$

- (b) We can obtain the system of [4] from (a) by making the following changes:
- Let  $f_i : \mathcal{S}_i \rightarrow \text{Ord}$  be defined as  $f_i$  in Example 2.7. (i) of [18], but with basis  $\mathbf{I}$ .
  - Restrict  $\psi_{\Omega_{\alpha+1}}(A)$  to  $A$  such that  $f_i(A) \in C_{\Omega_{\alpha+1}}(f_i(A))$ , where  $C_{\Omega_{\alpha+1}}$  is defined as in [4] (or equivalently such that  $\tilde{k}(A) \subseteq C_{\Omega_{\alpha+1}}(f_i(A))$ ).
  - Restrict  $\psi_{\mathbf{I}}(A)$  to  $A$  such that  $f_i(A) \in C_{\mathbf{I}}(f_i(A))$ .
  - Insert in  $(\text{Arg}_{\Omega_{\alpha+1}}, \prec'_{\Omega_{\alpha+1}})$  between  $\text{Arg}'_{\Omega_{\alpha+1}}$  and the “ $\psi_{\Omega_{\alpha+1}}$ -terms”  $\varphi_{]0, \Omega_{\alpha+1}[}([0, \Omega_{\alpha+1}[$ ), define  $k'_{\Omega_{\alpha+1}}, l'_{\Omega_{\alpha+1}}$  for these new terms as  $k, l$  in Example 2.7. (f) of [18] and  $\text{level}(\varphi_a b) := 0_l$ . (Note that we have already the  $\omega$ -function, which corresponds to  $\lambda\gamma.\varphi_0\gamma$  in the system of [4]).

Then one can easily see that we obtain again a cardinality based I-OFG ( $\kappa_0 = \mathbf{I}$ ),  $\text{eval}_{\Omega_{\alpha+1}}$  on  $\text{CNF}_{\Omega_{\alpha+1}}(\vec{a})$ , and  $\text{eval}_{\mathbf{I}}$  on  $\Omega_{\alpha}$  have the same values as in (a),  $\text{eval}_{\Omega_{\alpha+1}}(\varphi_{\alpha}\beta) = \varphi_{\alpha}\beta$ ,  $\text{eval}_{\Omega_{\alpha+1}}(\psi_{\Omega_{\alpha+1}}(A)) = \psi_{\Omega_{\alpha+1}}(f_i(A))$ ,  $\text{eval}_{\mathbf{I}}(\psi_{\mathbf{I}}(A)) = \psi_{\mathbf{I}}(f_i(A))$ , where  $\psi_{\Omega_{\alpha+1}}, \psi_{\mathbf{I}}$  are as in [4], and the term set of the underlying ordinal system together with its order is isomorphic to the set of terms in [4] which make use of terms  $\prec \omega_{\mathbf{I}}(\mathbf{I} + 1)$  only (where  $\omega_0(a) := a$ ,  $\omega_{\mathbf{I}+1}(a) := \omega^{\omega_{\mathbf{I}}(a)}$ ). Therefore in the limit the order types of these I-OS reach  $|\text{KPI}|$ .

**Remark 6.7.** *The (straightforward) generalization of Remark 4.7. in [18] to I-OFG holds.*

**Lemma 6.8.** (a) *For every OFG in the Example 6.6 there exists an elementary OS based on it.*

(b) *The supremum of the strength of I-OS is  $|\text{KPI}|$ .*

*Proof.* (a) as for Lemma 2.9. of [18]. (b) The elementary OS based on the sequence of OFG of Example 6.6 (b) (as well those of 6.6 (a)) have in the limit the strength of  $|\text{KPI}|$ .

**Theorem 6.9.** (a) *Every I-OS is well-ordered.*

- (b) Every elementary I-OS has order type below  $|KPI| = |(\Delta_2^1 - CA) + (BI)| = \psi(\epsilon_{1+1})$ .
- (c) The bound in (a) is sharp.

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