

Wheels

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Abstract

The following is not very well checked yet, i.e. there might be some typos and mistakes.

Definition 1 (Quotientwheel of an integral domain)

Let $\mathbb{R} := (|\mathbb{R}|, 0, 1, +, \cdot, -)$ be an integral domain (integral domain always with $0 \neq 1$). We write \mathbb{R} for $|\mathbb{R}|$.

- (a) Define \sim on \mathbb{R}^2 by $(a, b) \sim (a', b') \Leftrightarrow \exists r, r' \in \mathbb{R}. r \neq 0 \wedge r' \neq 0 \wedge (a \cdot r, b \cdot r) = (a' \cdot r', b' \cdot r')$.
- (b) Let $\frac{a}{b}$ be the equivalence class of (a, b) modulo \sim .
- (c) Let $\mathbb{R}_{\perp}^{\infty} := (|\mathbb{R}_{\perp}^{\infty}|, 0', 1', +', \cdot', -', (\frac{1}{\cdot})', \infty, \perp)$, the *quotient wheel of \mathbb{R}* (name preliminary), be defined by:
 - $|\mathbb{R}_{\perp}^{\infty}| = \{\frac{a}{b} \mid a, b \in \mathbb{R}\}$.
 - $\frac{a}{b} +' \frac{a'}{b'} := \frac{a \cdot b' + a' \cdot b}{b \cdot b'}$.
 - $\frac{a}{b} \cdot' \frac{a'}{b'} := \frac{a \cdot b}{a' \cdot b'}$.
 - $- ' \frac{a}{b} := \frac{-a}{b}$.
 - $(\frac{1}{\frac{a}{b}})' := \frac{b}{a}$.
 - $\infty := \frac{1}{0}$.
 - $\perp := \frac{0}{0}$.

*This article was inspired by discussions during Jens Blanck's lectures [Bla97] on [Pot97]. The idea to extend the quotient field by allowing fractions with denominator 0 is due to P. Martin-Löf.

- (d) We write \mathbb{R}_\perp^∞ instead of $|\mathbb{R}_\perp^\infty|$.
Let $\iota : \mathbb{R} \rightarrow \mathbb{R}_\perp^\infty$, $\iota(a) := \frac{a}{1}$. We identify a with $\iota(a)$, \mathbb{R} with $\iota[\mathbb{R}]$, and omit $'$ and sometimes \cdot . (Note $0' = \frac{0}{1} = \iota(0)$, $1' = \iota(1)$).

Lemma 2 (a) $\frac{a}{b} = \iota(a) \cdot (\frac{1}{\cdot})'(\iota(b))$.

(b) $\mathbb{R}_{\perp}^{\infty} = \text{Quot}(\mathbb{R})$ is the quotient field of \mathbb{R} , then $\mathbb{R}_{\perp}^{\infty} = \text{Quot}(\mathbb{R}) \cup \{\infty, \perp\}$.

(c) Let $a, b \in \text{Quot}(\mathbb{R})$. Then the following “multiplication tables” hold in $\mathbb{R}_{\perp}^{\infty}$:

$+$	a	∞	\perp
b	$a + b$	∞	\perp
∞	∞	\perp	\perp
\perp	\perp	\perp	\perp

\cdot	$a \neq 0$	0	∞	\perp
$b \neq 0$	ab	0	∞	\perp
0	0	0	\perp	\perp
∞	∞	\perp	∞	\perp
\perp	\perp	\perp	\perp	\perp

	$\frac{1}{x}$	$-x$
$a \neq 0$	$\frac{1}{a}$	$-a$
0	∞	0
∞	0	∞
\perp	\perp	\perp

Proof:

(a) $a \cdot (\frac{1}{\cdot})'(b) = \frac{a}{1} \cdot \frac{1}{b} = \frac{a}{b}$.

(b) Only \subset : If $b \neq 0$, then $\frac{a}{b} = \frac{a \cdot \frac{1}{b}}{b \cdot \frac{1}{b}} = \iota(a \cdot \frac{1}{b})$.

$\frac{a}{0} = \frac{1}{0} = \infty$ if $a \neq 0$.

$\frac{0}{0} = \perp$.

(c): Verify.

Definition 3 (a) A *wheel* is an n -tuple

$$\mathbb{R}_{\perp}^{\infty} = (|\mathbb{R}_{\perp}^{\infty}|, 0, 1, +, \cdot, -, \frac{1}{\cdot}, \infty, \perp) ,$$

where $|\mathbb{R}_{\perp}^{\infty}|$ is a set (written as $\mathbb{R}_{\perp}^{\infty}$), $0, 1, \infty, \perp \in \mathbb{R}_{\perp}^{\infty}$, $-, \frac{1}{\cdot}$ are unary functions on $\mathbb{R}_{\perp}^{\infty}$, and $+, \cdot$ are binary functions on $\mathbb{R}_{\perp}^{\infty}$, written in the usual way, such that for all $x, y, z \in \mathbb{R}_{\perp}^{\infty}$ the following axioms hold:

Commutativity	$x + y = y + x.$ $x \cdot y = y \cdot x.$
Associativity	$(x + y) + z = x + (y + z).$ $(x \cdot y) \cdot z = x \cdot (y \cdot z).$
Distributivity	$z \neq \infty \rightarrow (x + y) \cdot z = x \cdot z + y \cdot z,$
Neutral Elements	$x + 0 = x$ $x \cdot 1 = x.$
Inverse Elements	$x \notin \{\infty, \perp\} \rightarrow x + (-x) = 0.$ $x \notin \{0, \infty, \perp\} \rightarrow x \cdot \frac{1}{x} = 1.$
Definition of ∞, \perp	$\infty = \frac{1}{0}.$ $\perp = 0 \cdot \infty$
Strictness	$x \cdot \perp = \perp$ $x + \perp = \perp.$ $-\perp = \perp.$ $\frac{1}{\perp} = \perp.$
Laws for ∞	$x \notin \{\infty, \perp\} \rightarrow x + \infty = \infty.$ $\infty + \infty = \perp.$ $x \notin \{0, \perp\} \rightarrow x \cdot \infty = \infty.$ $-\infty = \infty.$ $\frac{1}{\infty} = 0.$
Non-triviality	$0 \neq 1.$

(b) $\frac{x}{y} = x \cdot \frac{1}{y},$
 $x - y := x + (-y).$

The usual conventions like omitting of \cdot , omitting parentheses (where \cdot binds more than $+$) apply.

Remark 4 *The following axioms*

$$\begin{aligned}
\text{Strictness} \quad & x \cdot \perp = \perp \\
& x + \perp = \perp. \\
& - \perp = \perp. \\
& \frac{1}{\perp} = \perp. \\
\text{Laws for } \infty \quad & x \notin \{\infty, \perp\} \rightarrow x + \infty = \infty. \\
& \infty + \infty = \perp. \\
& x \notin \{0, \perp\} \rightarrow x \cdot \infty = \infty. \\
& -\infty = \infty. \\
& \frac{1}{\infty} = 0.
\end{aligned}$$

can be replaced by the following axioms:

$$\begin{aligned}
\text{Laws of } \frac{1}{\cdot} \quad & \frac{1}{\frac{1}{x}} = x \\
& \frac{\frac{1}{x}}{xy} = \frac{1}{x} \frac{1}{y}. \\
& y \neq \infty \rightarrow \frac{x}{y} + z = \frac{x+zy}{y}. \\
\text{Laws for } - \quad & (-x)y = -(xy). \\
& \left(-\frac{1}{x}\right) = \frac{1}{-x}. \\
\text{Non triviality} \quad & \infty \neq 0.
\end{aligned}$$

or even the special cases:

$$\begin{aligned}
\text{Laws of } \frac{1}{\cdot} \quad & \frac{1}{\infty} = 0 \\
& \frac{1}{0 \cdot y} = \infty \cdot \frac{1}{y}. \\
& \frac{x}{0} + z = \frac{x+z \cdot 0}{0}. \\
\text{Axioms of } - \quad & - \perp = \perp. \\
& -\infty = \infty.
\end{aligned}$$

Lemma 5 *If R is an integral domain, its quotient wheel is a wheel.*

Proof:

$$\begin{aligned}
\frac{a}{b} + \frac{a'}{b'} &= \frac{ab' + a'b}{a'b'} = \frac{a'}{b'} + \frac{a}{b}. \\
\frac{a}{b} \cdot \frac{a'}{b'} &= \frac{aa'}{bb'} = \frac{a}{b} + \frac{a'}{b'}. \\
\left(\frac{a}{b} + \frac{c}{d}\right) + \frac{e}{f} &= \frac{ad+bc}{bd} + \frac{e}{f} = \frac{adf+bcf+bde}{bdf} = \frac{a}{b} + \frac{cf+de}{df} = \frac{a}{b} + \left(\frac{c}{d} + \frac{e}{f}\right). \\
\text{If } \frac{a}{b} \neq \infty, \text{ i.e. } f \neq 0 \vee e = 0, \text{ then } &\left(\frac{a}{b} + \frac{c}{d}\right) \cdot \frac{e}{f} = \frac{ad+bc}{bd} \cdot \frac{e}{f} = \frac{ade+bce}{bdf} = \\
\frac{adef+bcef}{bdf} &= \frac{ae}{bf} + \frac{ce}{df} = \frac{a}{b} \cdot \frac{e}{f} + \frac{c}{d} \cdot \frac{e}{f}.
\end{aligned}$$

The other laws are even more straight forward.

Task 6 *Minimize the number of axioms. (More precisely, the number of symbols for writing a complete axiomatization).*

The proofs for the following lemmata are not optimized yet.

Lemma 7 Assume $\mathbb{R}_{\perp}^{\infty}$ is a wheel as above. For $x, y \in \mathbb{R}_{\perp}^{\infty}$ we have

(a) $0, 1, \infty, \perp$ are distinct.

(b) $\mathbb{R}_{\perp}^{\infty} \setminus \{\perp, \infty\}$ is a field (note that $\frac{1}{\cdot}$ is then partial).

(c) If $x + y = x$, $x \notin \{\infty, \perp\}$, then $y = 0$.

(d) If $x \cdot y = x$, $x \notin \{0, \infty, \perp\}$ then $y = 1$.

(e) If $x + y = 0$, then $y = -x$.

(f) $x \cdot 0 = 0$, if $x \notin \{\infty, \perp\}$.

(g) If $x \cdot y = 1$, then $y = \frac{1}{x}$.

(h) $-0 = 0$.

(i) $\frac{1}{1} = 1$.

(j) $x + y = \perp \iff x = \perp \vee y = \perp \vee x = y = \infty$.

(k) If $x + y \in \{0, \perp\}$ then $(x + y) \cdot \infty = x \cdot \infty + y \cdot \infty$.
Otherwise $(x + y) \cdot \infty = \infty \neq \perp = x \cdot \infty + y \cdot \infty$.

(l) $(-x)y = -(xy) = x(-y)$

(m) $-(-x) = x$.

(n) $-$ is injective.

(o) $\frac{1}{\cdot}$ is injective.

(p) $\frac{1}{x} \cdot \frac{1}{y} = \frac{1}{xy}$.

(q) $\frac{1}{\frac{1}{x}} = x$.

(r) $\frac{1}{-x} = -\frac{1}{x}$.

(s) $y \neq \infty \rightarrow \frac{x}{y} + w = \frac{x+wy}{y}$

(t) $y, y' \notin \{\infty, \perp\} \rightarrow \frac{x}{y} + \frac{x'}{y'} = \frac{xy' + x'y}{yy'}$

Proof:

(a) If $\perp = 1$, then $\forall x. x = x \cdot 1 = x \cdot \perp = \perp$, $0 = 1$, contradiction.

If $\perp = 0$, then $\forall x. x = x + 0 = x + \perp = \perp$, $0 = 1$, contradiction.

If $\perp = \infty$, then $0 = \frac{1}{\infty} = \frac{1}{\perp} = \perp$, contradiction.

If $\infty = 0$, then $1 = 1 + 0 = 1 + \infty = \infty = 0$, contradiction.

If $\infty = 1$, then $\perp = 0 \cdot \infty = 0 \cdot 1 = 0$, contradiction.

$0 \neq 1$.

(b) by (a).

The remaining assertions follow now from (b). But let's do it from scratch:

(c) $y = x + y - x = x - x = 0$.

(d) $y = \frac{1}{x}xy = \frac{1}{x}x = 1$.

(e) If $x \notin \{\infty, \perp\}$, then $y = -x + x + y = -x + x = 0$.

If $x = \perp$, $x + y = \perp \neq 0$.

If $x = \infty$, $x + y \in \{\infty, \perp\}$, $x + y \neq 0$.

(f) $x \cdot 0 + x = x \cdot 0 + x \cdot 1 = x \cdot (0 + 1) = x \cdot 1 = x$, $x \cdot 0 = 0$.

(g) If $x \notin \{0, \infty, \perp\}$, then $y = \frac{1}{x}xy = \frac{1}{x}x = 1$.

If $x = 0$, $xy \in \{0, \perp\}$, $xy \neq 1$.

If $x = \perp$, $x \cdot y = \perp \neq 0$.

If $x = \infty$, $x \cdot y \in \{\infty, \perp\}$, $x \cdot y \neq 0$.

(h): $0 + -0 = 0$, $-0 = 0$.

(i): $\frac{1}{1} \cdot 1 = 1$, $1 = \frac{1}{1}$.

(j): " \leftarrow " trivial. " \rightarrow ": Assume right side is false, left side holds.

If $x = \infty$, then $x + y = \infty \neq \perp$.

Otherwise $y = -x + (x + y) = -x + \perp = \perp$, contradicting the falsity of the right side.

(k): Case $x + y = 0$. Then $x \neq \perp$, $(x + y) \cdot \infty = 0 \cdot \infty = \perp$.

Subcase $x = y = 0$: $x \cdot \infty + y \cdot \infty = \perp$.

Otherwise $x, y \notin \{0, \infty, \perp\}$, $x = -y$, $x \cdot \infty + y \cdot \infty = \infty + \infty = \perp$.

Case $x + y = \perp$. Again $(x + y) \cdot z = \perp$.

Case $x = \perp \vee y = \perp$. Right side is \perp .

Otherwise follows $x = y = \infty$, right side is \perp .

Case $x + y \notin \{0, \perp\}$. Then $(x + y) \cdot \infty = \infty$.

$x \neq \perp$, $y \neq \perp$.

If $x = 0$, then $x \cdot \infty + y \cdot \infty = \perp$.

If $y = 0$, the assertion follows again.

Otherwise $x \cdot \infty + z \cdot \infty = \infty + \infty = \perp$.

(l) If $x \notin \{\infty, \perp\}$, then $(-x)y + xy = (-x + x)y = 0 \cdot y = 0$, $-(xy) = (-x)y$.

If $x = \infty, \perp$, the assertion is an axiom.

(m) $x = \infty, \perp$: by the axioms. Otherwise: $x + (-x) = 0$, therefore $x = -(-x)$.

(n) Assume $-x = -y$.

Case $x = \perp$. $y \neq \infty$. If $y \neq \perp$, then $0 = y + -y = y + \perp = \perp$, contradiction.

Case $x = \infty$: $y \neq \perp$. If $y \neq \infty$, then $0 = y + -y = \infty$, contradiction.

Case $y = \perp$, $y = \infty$: similar.

Otherwise $x = x + -y + y = x + -x + y = y$. (o) Assume $\frac{1}{x} = \frac{1}{y}$.

If $x = \perp$, then $\frac{1}{y} = \perp$, $y \notin \{0, \infty\}$. If $y \neq \perp$, $1 = y \cdot \frac{1}{y} = \perp$, contradiction.

If $x = 0$, then $y \notin \{\perp, \infty\}$. If $y \neq 0$, then $1 = y \cdot \frac{1}{y} = \infty$, contradiction.

If $y = \perp, 0$, follows again the assertion.

Otherwise $x = x \frac{1}{y} y = x \frac{1}{x} y = y$. (p) $x, y \notin \{0, \infty, \perp\}$: $xy \frac{1}{x} \frac{1}{y} = 1$.

$x = \perp$: $\frac{1}{x} \cdot \frac{1}{y} = \perp = \frac{1}{xy}$.

$y = \perp$ similarly.

Assume $x, y \neq \perp$.

Case $x = 0$, $y \neq \infty$: $\frac{1}{y} \notin \{0, \perp\}$. $\frac{1}{xy} = \frac{1}{0} = \infty$, $\frac{1}{x} \frac{1}{y} = \infty \frac{1}{y} = \infty$.

Case $x = 0$, $y = \infty$: both sides are \perp .

Case $x = \infty$, $y = 0$: similarly.

Case $x = \infty$, $y \neq 0$: $\frac{1}{xy} = 0 = \frac{1}{x} \frac{1}{y}$.

Case $y = \infty$: similarly.

(q): $x = 0, \infty, \perp$: trivial. Otherwise $\frac{1}{x} \cdot x = 1$, $\frac{1}{\frac{1}{x}} = x$.

(r): The cases $x = 0, \infty, \perp$ can be easily verified. Otherwise we have $(-x) \cdot (-\frac{1}{x}) = -(-(x \cdot \frac{1}{x})) = x$, and the assertion.

(s): If $x = \perp$ or $y = \perp$ or $w = \perp$, both sides are \perp . Assume $x, y, w \neq \perp$.

Case $x = y = 0$: $wy \in \{0, \perp\}$, left side is \perp , right side is $\frac{0}{0}$ or $\frac{\perp}{0}$, therefore \perp .

Case $y = 0$, $w = \infty$: both sides \perp .

Case $y = 0$, $w \neq \infty$: both sides are \perp , if $x = 0$ and ∞ otherwise.

Otherwise: $y \neq 0, \infty, \perp$, $\frac{x}{y} + w = \frac{1}{y} \cdot y \cdot (\frac{x}{y} + w) = \frac{1}{y} \cdot (x + wy)$.

(t): By (s)

Lemma 8 *Let $a, b, c, d, e, f, r, s \in \mathbb{R}_{\perp}^{\infty} \setminus \{\infty, \perp\}$ (as well with indices and accents), $\mathbb{R}_{\perp}^{\infty}$ be a wheel.*

(a) *Every element of $\mathbb{R}_{\perp}^{\infty}$ can be written as $\frac{x}{y}$ with $x, y \neq \perp, \infty$.*

(b) $\frac{a}{b} = \frac{a'}{b'} \iff \exists r \notin \{0, \infty, \perp\}. (ar, br) = (a'r', b'r')$.

(c) $\frac{a}{b} + \frac{a'}{b'} = \frac{ab' + a'b}{bb'}$.

(d) $-\frac{a}{b} = \frac{-a}{b}$.

$$(e) \frac{a}{b} \frac{c}{d} = \frac{ac}{bd}.$$

$$(f) \frac{1}{\frac{a}{b}} = \frac{b}{a}.$$

Proof:

(a) $\infty = \frac{1}{0}$, $\perp = \frac{0}{0}$. Otherwise $x = \frac{x}{1}$.

(b) Assume $\frac{a}{b} = \frac{a'}{b'}$. Case $b, b' \neq 0$: $ab' = \frac{a}{b}bb' = \frac{a'}{b'} = a'b$, choose $r = b'$, $r' = b$. If $b = 0 \vee b' = 0$, then $\frac{a'}{b'} = \frac{a}{b} \in \{\infty, \perp\}$, $b = b' = 0$. Further $a = 0 \iff \frac{a}{b} = \perp \iff a' = 0$. Choose, if $a = 0$, $r, r' = 1$ otherwise $r := a'$, $r' := a$.

In the other direction it suffices to show $\frac{a}{b} = \frac{ar}{br}$ if $a, b, r \notin \{\infty, \perp\}$, $r \neq 0$.

If $b = 0$, $br = 0$, further $a = 0 \iff ar = 0$ (multiplb with $\frac{1}{r}$), $\frac{a}{b} = \frac{ar}{br}$.

Otherwise $\frac{ar}{br}b = \frac{1}{r}rb\frac{ar}{br} = ar\frac{1}{r} = a(r\frac{1}{r}) = a$, $\frac{a}{b} = a\frac{1}{b} = \frac{ar}{br}$.

(c), (d), (e), (f): Immediate by the properties proved before.

References

- [Bla97] J. Blanck. One possible approach to efficient implementation of exact real arithmetic (I – III). Lectures given in the Logic seminar Uppsala-Stockholm, 1997.
- [Pot97] P. J. Potts. Efficient on-line computation of real functions using exact floating point. Manuscript, Dept. of Computing, Imperial College, London, 1997.