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II.4.1.1. Regular Grammars and NFAs (13.5)

II.4.1.2. Translating NFAs into Regular Expressions (13.10)

II.4.1.3. Main Theorem
Theorem II.4.1.1

Theorem (II.4.1.1)

For every right linear grammar $G$ there exists an NFA $A$ s.t.

$$L(G) = L(A)$$

A can be computed from $G$. 
II.4.1. Equivalence Theorem for Regular Languages

II.4.1.1. Regular Grammars and NFAs (13.5)

Proof of Theorem II.4.1.1

We show that \( L(A) = L(G) \):

- Assume \( w = a_1 \cdots a_n \in L(A) \).

  Then there exists a sequence of transitions in \( A \)

  \[
  S = A_0 \xrightarrow{a_1} A_1 \xrightarrow{a_2} A_2 \xrightarrow{a_3} \cdots \xrightarrow{a_{n-1}} A_{n-1} \xrightarrow{a_n} q_F
  \]

  or

  \[
  S = A_0 \xrightarrow{a_1} A_1 \xrightarrow{a_2} A_2 \xrightarrow{a_3} \cdots \xrightarrow{a_n} A_n \in F
  \]

  But from this we obtain derivations

  \[
  S = A_0 \rightarrow a_1 A_1 \rightarrow a_1 a_2 A_2 \rightarrow \cdots \rightarrow a_1 a_2 \cdots a_{n-1} A_{n-1} \\
  \rightarrow a_1 a_2 \cdots a_{n-1} a_n = w
  \]

  or

  \[
  S = A_0 \rightarrow a_1 A_1 \rightarrow a_1 a_2 A_2 \rightarrow \cdots \rightarrow a_1 a_2 \cdots a_n A_n \\
  \rightarrow a_1 a_2 \cdots a_n = w
  \]

  So \( w \in L(G) \).
Proof of Theorem II.4.1.1

Assume \( w = a_1 \cdots a_n \in L(G) \).
A derivation will have the form

\[
S = A_0 \rightarrow a_1A_1 \rightarrow a_1a_2A_2 \rightarrow \cdots \rightarrow a_1a_2 \cdots a_{n-1}A_{n-1} \\
\rightarrow a_1a_2 \cdots a_{n-1}a_n = w
\]

or

\[
S = A_0 \rightarrow a_1A_1 \rightarrow a_1a_2A_2 \rightarrow \cdots \rightarrow a_1a_2 \cdots a_nA_n \\
\rightarrow a_1a_2 \cdots a_n = w
\]

Then there exists a sequence of transitions in \( A \)

\[
S = A_0 \xrightarrow{a_1} A_1 \xrightarrow{a_2} A_2 \xrightarrow{a_3} \cdots \xrightarrow{a_{n-1}} A_{n-1} \xrightarrow{a_n} q_F
\]

or

\[
S = A_0 \xrightarrow{a_1} A_1 \xrightarrow{a_2} A_2 \xrightarrow{a_3} \cdots \xrightarrow{a_n} A_n \in F
\]

So \( w \in L(A) \).
II.4.1. Equivalence Theorem for Regular Languages

II.4.1.1. Regular Grammars and NFAs (13.5)

II.4.1.2. Translating NFAs into Regular Expressions (13.10)

II.4.1.3. Main Theorem
Theorem II.4.1.2

Let $A = (Q, q_0, F, T, \rightarrow)$ be an NFA.
Then there exist a regular expression $E$ s.t. $L(E) = L(A)$.
$E$ can be computed from $A$. 
Example

Before proving Theorem II.4.1.2 we give an example:
Consider the following automaton for the language $L = \ast$.

![Automaton Diagram]

We define regular expressions and simplify them at each intermediate step in order to keep them simple.
From $A$ to $E^\emptyset_{q,q'}$

Original automaton:

Let $L^\emptyset_{q,q'}$ be the set of strings which allows us to get from $q$ to $q'$ with intermediate states in $\emptyset$, i.e. without any intermediate states. We define a regular expression $E^\emptyset_{q,q'}$, s.t. $L(E^\emptyset_{q,q'}) = L^\emptyset_{q,q'}$. We can define:

- $E^\emptyset_{q,q'} := a_1 \mid \cdots \mid a_n$, 
  if $q \neq q'$ and we have transitions $q \xrightarrow{a_i} q'$,

- $E^\emptyset_{q,q'} = a_1 \mid \cdots \mid a_n \mid \epsilon$, 
  if $q = q'$ and we have transitions $q \xrightarrow{a_i} q'$.
Calculation of $L_{q,q'}^\emptyset$

Original automaton:

\[
\begin{align*}
E_{q_0,q_0}^\emptyset & = 1 \mid \epsilon \\
E_{q_0,q_1}^\emptyset & = 0 \\
E_{q_1,q_0}^\emptyset & = \emptyset \\
E_{q_1,q_1}^\emptyset & = 0 \mid 1 \mid \epsilon
\end{align*}
\]
From $A$ to $L_{q,q'}^{\emptyset}$

Original automaton:

States with $E_{q,q'}^{\emptyset}$:
From $E_{q,q'}^\emptyset$ to $E_{q,q'}^{q_0}$

Let $L_{q,q'}^{q_0}$ be the set of strings which allows us to get from $q$ to $q'$ with intermediate states in $\{q_0\}$.

We define $E_{q,q'}^{q_0}$ s.t. $L(E_{q,q'}^{q_0}) = L_{q,q'}^{q_0}$:

$$E_{q,q'}^{q_0} = E_{q,q'}^\emptyset | (E_{q,q_0}^\emptyset (E_{q_0,q_0}^\emptyset)^* E_{q_0,q'}^\emptyset)$$
Calculation of $E_{q, q'}^{q_0}$

$E_{q, q'}^{q_0} = E_{q, q'}^{q_0} \mid (E_{q, q_0}^{q_0} (E_{q_0, q_0}^{q_0})^* E_{q_0, q'}^{q_0})$:

\[E_{q_0, q_0}^{q_0} = (1 \mid \varepsilon) \mid ((1 \mid \varepsilon)(1 \mid \varepsilon)^*(1 \mid \varepsilon)) = 1^*\]

\[E_{q_0, q_1}^{q_0} = 0 \mid ((1 \mid \varepsilon)(1 \mid \varepsilon)^*0) = 1^*0\]

\[E_{q_1, q_0}^{q_0} = \emptyset \mid (\emptyset (1 \mid \varepsilon)^*0) = \emptyset\]

\[E_{q_1, q_1}^{q_0} = (0 \mid 1 \mid \varepsilon) \mid (\emptyset (1 \mid \varepsilon)^*0) = 0 \mid 1 \mid \varepsilon\]
II.4.1. Equivalence Theorem for Regular Languages

II.4.1.2. Translating NFAs into Regular Expressions (13.10)

From $E_{q,q'}^\emptyset$ to $E_{q,q'}^{q_0}$

States with $E_{q,q'}^\emptyset$:

States with $E_{q,q'}^{q_0}$:
From $E_{q,q'}^{q_0}$ to $E_{q,q'}^{q_0,q_1}$

Let $L_{q,q'}^{q_0,q_1}$ be the set of strings which allows us to get from $q$ to $q'$ with intermediate states in \{q_0, q_1\}.

We define $E_{q,q'}^{q_0,q_1}$, s.t. $L(E_{q,q'}^{q_0,q_1}) = L_{q,q'}^{q_0,q_1}$:

$$E_{q,q'}^{q_0,q_1} = E_{q,q'}^{q_0} \mid (E_{q,q_1}^{q_0}(E_{q_1,q_1}^{q_0})^* E_{q_1,q'}^{q_0})$$
Calculation of $E_{q_0,q_1}^{q_0,q_1}$

$E_{q_0,q_1}^{q_0,q_0} = 1^* | (1^*0(0|1|\epsilon)^*\emptyset)$
$= 1^*$

$E_{q_0,q_1}^{q_0,q_1} = (1^*0) | (1^*0(0|1|\epsilon)^*(0|1|\epsilon))$
$= 1^*0(0|1)^*$

$E_{q_0,q_1}^{q_1,q_0} = \emptyset | ((0|1|\epsilon)(0|1|\epsilon)^*\emptyset$
$= \emptyset$

$E_{q_0,q_1}^{q_1,q_1} = (0|1|\epsilon) | ((0|1|\epsilon)(0|1|\epsilon)^*(0|1|\epsilon))$
$= (0|1)^*$
From $E_{q_0, q'}^{q_0}$ to $E_{q, q'}^{q_0, q_1}$

States with $E_{q_0, q'}^{q_0}$:

States with $E_{q, q'}^{q_0, q_1}$, the complete language between those states:
The Language of $A$: $L(A)$

States with $E_{q_0,q_1}^{q_0,q_1}$:

- $L(E_{q_0,q_1}^{q_0,q_1})$ is the set of strings which allows us to get from $q_0$ to $q_1$ using any intermediate states.
- The language $L(A)$ is the set of strings which allow us to get from $q_0$ to any accepting state.
- In the example there is only one accepting state ($q_1$), so the language accepted by $A$ is the language given by

$$E_{q_0,q_1}^{q_0,q_1} = 1^*0(0 \mid 1)^*$$
The Language of $A$: $L(A)$

States with $E^{q_0,q_1}_{q,q'}$:

Let $A'$ be as $A$, but with additional accepting state $q_0$, then we get that $L(A')$ is given by

$$E^{q_0,q_1}_{q_0,q_0} \mid E^{q_0,q_1}_{q_0,q_1} = 1^* \mid (1^*0(0 \mid 1)^*) = (0 \mid 1)^*$$
Proof of Theorem II.4.1.2

Let for states $q, q'$ of $A$

$$L_{q,q'} := \{ w \in T^* \mid q \xrightarrow{w} q' \}$$

We construct for states $q, q'$ of $A$ a regular expression $E_{q,q'}$ s.t.

$$L(E_{q,q'}) = L_{q,q'}$$

If $F = \{q_1, \ldots, q_k\}$ then we obtain

$$L(A) = L_{q_0,q_1} \mid \cdots \mid L_{q_0,q_k} = L(E_{q_0,q_1} \mid E_{q_0,q_2} \mid \cdots \mid E_{q_0,q_k})$$

(If $F$ is empty, then $L(A) = L(\emptyset)$).
Proof of Theorem II.4.1.2

We define regular expressions $E_{q,q'}$ in stages by referring to $E_{q_1,\ldots,q_l}$, s.t.

$$L(E_{q_1,\ldots,q_l}) = L_{q_1,\ldots,q_l} := \{a_1 \cdots a_k \in T^* \mid \exists p_i \in \{q_1,\ldots,q_l\}. q \xrightarrow{a_1} p_1 \xrightarrow{a_2} p_2 \xrightarrow{q_3} \cdots \xrightarrow{a_{k-1}} p_{k-1} \xrightarrow{a_k} q' \}$$

So $L_{q,q'}$ is the set of words which allow us to get from $q$ to $q'$ by using as intermediate states $q_1,\ldots,q_l$ only.

We define $E_{q_1,\ldots,q_k}$ by induction on $k$.

Then we can define $E_{q,q'} := E_{q,q'}^Q$. 
Proof of Theorem II.4.1.2

Base case $k = 0$:
Let $a_1, \ldots, a_k$ be the $a_i$ s.t. $q \xrightarrow{a_i} q'$. Then

$$E_{q, q'}^{\emptyset} := \begin{cases} a_1 | \cdots | a_k & \text{if } q \neq q' \\ a_1 | \cdots | a_k | \epsilon & \text{if } q = q' \end{cases}$$

(in case of $k = 0$ we have $E_{q, q'}^{\emptyset} = \emptyset$ or = $\epsilon$).
Proof of Theorem II.4.1.2

Induction Step: Assume we have defined \( E_{q_1,\ldots,q_{k-1}}^{p,p'} \) for all \( p, p' \in Q \).

We define \( E_{q,q'}^{q_1,\ldots,q_{k-1}} \).

A transition \( q \xrightarrow{w} q' \) which uses only intermediate states \( q_1,\ldots,q_k \) can have two forms:

- Either we don’t use \( q_k \) as an intermediate state. So we have only intermediate states \( q_1,\ldots,q_{k-1} \) and have \( w \in L_{q,q'}^{q_1,\ldots,q_{k-1}} \).

- Or we reach \( q_k \) as an intermediate state. We single out
  - the first part of the transition which doesn’t use state \( q_k \) until one reaches for the first time as an intermediate state \( q_k \) (note that \( q = q_k \) or \( q' = q_k \) is possible)
  - the second part where we several times go from \( q_k \) to \( q_k \) with intermediate states \( \neq q_k \),
  - and the last part where we get from \( q_k \) to \( q' \) without using \( q_k \).
So we have

\[ q \xrightarrow{v} q_k \xrightarrow{w_1} q_k \xrightarrow{w_2} q_k \xrightarrow{w_2} \cdots \xrightarrow{w_j} q_k \xrightarrow{v'} q' \]

where \( j = 0 \) is possible, all intermediate transitions avoid \( q_k \) and \( w = vw_1w_2\cdots w_kv' \).
Proof of Theorem II.4.1.2

In the second part we have

- \( v \in L_{q, q_k}^{q_1, \ldots, q_{k-1}} \),
- \( w_i \in L_{q_k, q_k}^{q_1, \ldots, q_{k-1}} \),
- \( v' \in L_{q_k, q_k}^{q_1, \ldots, q_{k-1}} \).

Therefore \( w = vw_1 \cdots w_k v' \in L_{q, q_k}^{q_1, \ldots, q_{k-1}} \cdot (L_{q_k, q_k}^{q_1, \ldots, q_{k-1}})^* \cdot L_{q_k, q_k}^{q_1, \ldots, q_{k-1}} \).

Therefore

\[
L_{q, q_k}^{q_1, \ldots, q_k} \subseteq L_{q, q_k}^{q_1, \ldots, q_{k-1}} \mid (L_{q_k, q_k}^{q_1, \ldots, q_{k-1}})^* \cdot L_{q_k, q_k}^{q_1, \ldots, q_{k-1}}
\]

One can see easily as well that for an element \( w \) in the right hand side we can derive that \( w \) is in the left hand side as well, i.e.

\[
L_{q, q_k}^{q_1, \ldots, q_k} \supseteq L_{q, q_k}^{q_1, \ldots, q_{k-1}} \mid (L_{q_k, q_k}^{q_1, \ldots, q_{k-1}})^* \cdot L_{q_k, q_k}^{q_1, \ldots, q_{k-1}}
\]
Proof of Theorem II.4.1.2

So

\[ L_{q_1, \ldots, q_k} = L_{q_1, \ldots, q_{k-1}} \mid (L_{q, q_k} \cdot (L_{q_k, q_k})^* \cdot L_{q_k, q'}) \]

and we can define

\[ E_{q_1, \ldots, q_k} = E_{q_1, \ldots, q_{k-1}} \mid (E_{q, q_k} \cdot (E_{q_k, q_k})^* \cdot E_{q_k, q'}) \]
II.4.1. Equivalence Theorem for Regular Languages

II.4.1.1. Regular Grammars and NFAs (13.5)
II.4.1.2. Translating NFAs into Regular Expressions (13.10)
II.4.1.3. Main Theorem
Theorem II.4.1.3

Let $L$ be a language over an alphabet $T$. The following are equivalent:

1. $L$ is definable by a regular expression.
2. $L$ is a regular.
3. $L$ is definable by a right-linear grammar.
4. $L$ is definable by a left-linear grammar.
5. $L$ is definable by an NFA with empty moves.
6. $L$ is definable by an NFA.
7. $L$ is definable by a DFA.

Furthermore, the corresponding regular expressions, right linear grammars, left-linear grammars, NFAs with empty moves, NFAs, DFAs can be computed from each other.
Proof of Theorem II.4.1.3

We are going to show that

- languages definable by regular expressions,
- languages definable by regular grammars,
- languages definable by NFAs with empty moves,
- languages definable by NFAs,
- languages definable by DFAs

all define the same language.

We already have everything in order to show the above with regular grammars restricted to right-linear grammars.
Lemma II.4.1.4

Let $L$ be a language over an alphabet $T$. The following is equivalent:

1. $L$ is definable by a regular expression.
2. $L$ is definable by a right-linear grammar.
3. $L$ is definable by an NFA with empty moves.
4. $L$ is definable by an NFA.
5. $L$ is definable by a DFA.

Furthermore, the corresponding regular expressions, right linear grammars, NFAs with empty moves, NFAs, DFAs can be computed from each other.
Proof of Lemma II.4.1.4

- (1) $\rightarrow$ (2) was shown in II.2.2.2.
  - (Finite languages are definable regular grammars. Languages definable by regular grammars are closed under the operations for forming regular expressions).
- (2) $\rightarrow$ (4) was shown in Theorem II.4.1.1
  - Right-linear grammars can be simulated by an NFA.
- (4) $\rightarrow$ (1) was shown in Theorem II.4.1.2
  - We can determine the language between states of an NFA as a regular expression.
- So (1), (2), (4) are equivalent.
II.4.1.3. Main Theorem

Proof of Lemma II.4.1.4

- (3) $\rightarrow$ (4) was shown in Theorem II.3.4.1.
  - We can omit the empty moves in NFAs with empty moves.
- (4) $\rightarrow$ (5) was shown in Theorem II.3.5.1.
  - NFAs can be translated into DFAs using as states sets of states.
- (5) $\rightarrow$ (4) $\rightarrow$ (3) are trivial.
  - DFAs are special cases of NFAs,
    NFAs are special cases of NFAs with empty moves.
- So (3), (4), (5) are equivalent.
- So (1), (2), (3), (4), (5) are equivalent.
Equivalence of Left-Linear and Right-Linear Grammars

It remains to show that left-linear and right-linear grammars are equivalent.

This is shown as follows:

- The languages derived by left-linear grammars are $L^R$ for languages derived by right-linear grammars.
- Regular Expressions are closed under the reverse operation, i.e. if $L$ is definable by a regular expression, so is $L^R$.
- Therefore as well right-linear grammars are closed under the reverse operation, therefore left-linear and right-linear grammars are equivalent.
Right-Linear Languages are the Reverse of Left-Linear Ones

Lemma (II.4.1.5)

1. Let $G$ be a left-linear grammar.
   Then there exist a right-linear grammar $G'$ over the same alphabet s.t. $L(G) = L(G')^R$.
   $G'$ can be computed from $G$.

2. Let $G$ be a right-linear grammar.
   Then there exist a left-linear grammar $G'$ over the same alphabet s.t. $L(G) = L(G')^R$.
   $G'$ can be computed from $G$. 
Proof of Lemma II.4.1.5

We prove only (1), (2) is analogously.
Let \( G \) be a left-linear grammar with alphabet \( T \), nonterminals \( N \) and start symbol \( S \).
Let \( G' \) be identical to \( G \) but with rules

\[
B \rightarrow aC
\]

\((B, C \in N, a \in T)\) replaced by

\[
B \rightarrow Ca
\]

\(G'\) is right-linear. Further it follows immediately for any \( w \in (N \cup T)^* \) that

\[
S \Rightarrow_G w \text{ iff } S \Rightarrow_{G'} w^R
\]
II.4.1. Equivalence Theorem for Regular Languages

II.4.1.3. Main Theorem

Proof of Lemma II.4.1.5

Therefore

\[ L(G') = \{ w \in T^* \mid S \Rightarrow_{G'} w \} \]
\[ = \{ w^R \in T^* \mid S \Rightarrow_G w \} \]
\[ = \{ w \in T^* \mid S \Rightarrow_G w \}^R \]
\[ = L(G)^R \]
Lemma (II.4.1.6)

1. For every regular expression $E$ there exists a regular expression $E^R$ s.t. $L(E^R) = L(E)^R$. $E^R$ can be computed from $E$.

2. Similarly for every language $L$ definable by a right-linear grammar $G$ there exists a right-linear grammar $G^R$ defining $L^R$. $G^R$ can be computed from $G$. 
(1) We show the existence of $E^R$ by induction on $E$:

- For $E = \emptyset$, $E = \epsilon$ or $E = a$, $L(E)^R = L(E)$, so define $E^R := E$.
- For $E = E_1 \mid E_2$ we have define $E^R = E_1^R \mid E_2^R$.
- For $E = E_1 E_2$ define $E^R = E_2^R E_1^R$.
- For $E = E_1^*$ define $E^R = (E_1^R)^*$.

(2) Follows since languages definable by right-linear grammars are exactly the languages definable by regular expressions.
Left-Linear and Right-Linear Grammars are Equivalent

Lemma (II.4.1.7)

Let $L$ be a language over an alphabet $T$. The following are equivalent:

1. $L = L(G)$ for a left-linear grammar $G$.
2. $L = L(G)$ for a right-linear grammar $G$.

The left-linear and right-linear grammars can be computed from each other.
Proof of Lemma II.4.1.7

- Assume $L = L(G)$ for a left-linear grammar $G$.
  - Then $L^R = L(G')$ for a right-linear grammar $G'$.
  - Right-linear grammars are closed under $L \mapsto L^R$.
  - Therefore there exists a right-linear grammar $G''$ s.t.
    \[ L(G'') = L(G')^R = (L^R)^R = L. \]

- Assume $L = L(G)$ for a right-linear grammar $G$.
  - There exists a right-linear grammar $G'$ s.t. $L(G') = L^R$.
  - There exists a left-linear grammar $G''$ s.t. $L(G'') = L(G')^R$.
  - Now $L(G'') = L(G')^R = (L(G)^R)^R = L(G) = L.$
Proof of Theorem II.4.1.3

By the above.