CS_275 Automata and Formal Language Theory

Course Notes
Additional Material
(This material is no longer taught and not exam relevant)
Part II: The Recognition Problem (II)
Sect II.5.: Context Free Grammars and Programming Languages (14)

Anton Setzer
(Based on a book draft by J. V. Tucker and K. Stephenson)
Dept. of Computer Science, Swansea University

http://www.cs.swan.ac.uk/∼csetzer/lectures/automataFormalLanguage/current/index.html

March 21, 2017

Notations for Derivation Trees

We introduce the following notations:

- If D is a derivation tree, then
  - $\mathrm{label}(D)$ is the label of the root of D.
  - $\mathrm{children}(D) = [D_1, \ldots, D_n]$ means that the children (immediate subtrees of D) are $D_1, \ldots, D_n$ (read from left to right).
  - $\mathrm{frontier}(D)$ denotes the frontier of D.
- If $D_1, \ldots, D_n$ are derivation trees, A a nonterminal, let $D := \mathrm{tree}(A, D_1, \ldots, D_n)$ be the derivation tree s.t.
  - $\mathrm{label}(D) = A$,
  - $\mathrm{children}(D) = [D_1, \ldots, D_n]$.
- A special derivation tree is the tree with only one node, namely the root and no subtrees for this node. It is called the **trivial derivation tree**.

Inductive Definition of Derivations

We can define the set of derivation trees as well inductively as follows:

Definition

We define the set of derivation trees D for a CFG $G = (T, N, S, P)$ inductively together with

- $\mathrm{label}(D) \in T \cup N \cup \{\epsilon\}$,
- $\mathrm{children}(D)$, a list of derivation trees,
- $\mathrm{frontier}(D) \in (T \cup N)^*$.

as follows:

- If $A \in T \cup N \cup \{\epsilon\}$, then
  - $D := \mathrm{tree}(A)$ is a derivation tree, (the **trivial derivation tree**),
  - $\mathrm{label}(D) := A$,
  - $\mathrm{children}(D) = [\,]$,
  - $\mathrm{frontier}(D) := A$.

Inductive Definition of Derivations

We can define the set of derivation trees as well inductively as follows:

Definition (Cont)

- *If $A \in N$, $D_1, \ldots, D_n$ are derivation trees,*

  $$A \to \mathrm{label}(D_1).\mathrm{label}(D_2).\cdots.\mathrm{label}(D_n) \textit{ is a production}$$

  and $n = 1 \vee \forall i.\mathrm{label}(D_i) \neq \epsilon$, then
  - *$D := \mathrm{tree}(A, D_1, \ldots, D_n)$ is a derivation tree,*
  - *$\mathrm{label}(D) := A$,*
  - *$\mathrm{children}(D) := [D_1, \ldots, D_n]$,*
  - *$\mathrm{frontier}(D) := \mathrm{frontier}(D_1).\mathrm{frontier}(D_2).\cdots.\mathrm{frontier}(D_n)$.*

Size of a Derivation Tree

We want show that for every derivation tree we can find a corresponding derivation and vice versa. For this we need a measure of the size of derivation tree.

Definition

The size $\mathrm{size}(D)$ of a derivation tree D is defined by induction on the definition of trees:

- If $D = \mathrm{tree}(x)$, then $\mathrm{size}(D) := 1$.
- If $D = \mathrm{tree}(A, D_1, \ldots, D_n)$, then $\mathrm{size}(D) := 1 + \mathrm{size}(D_1) + \cdots + \mathrm{size}(D_n)$.

We define as well the height $\mathrm{height}(D)$ of a derivation tree:

- If $D = \mathrm{tree}(x)$, then $\mathrm{height}(D) := 1$.
- If $D = \mathrm{tree}(A, D_1, \ldots, D_n)$, then $\mathrm{height}(D) := 1 + \max\{\mathrm{size}(D_1), \ldots, \mathrm{size}(D_n)\}$.

Proof of Theorem II.5.1.1.

Theorem II.5.1.1. follows by Lemma II.5.1.2. below.
We first introduce the notion of a derivation forest, which generalises derivation trees.

Derivation Forest

For proving the equivalence of derivation trees and derivations, we need to deal with derivations $w \Rightarrow^* w'$ where w is a string. Such derivations correspond to derivation forests, as defined as follows:

Definition

Let $G = (T, N, S, P)$ be a CFG, $w, w' \in (T \cup N)^*$, $w \neq \epsilon$. Let $w = x_1, \ldots, x_l$, $x_i \in T \cup N$.
A **derivation forest** with **root** w and frontier w' is a list of derivations $[D_1, \ldots, D_l]$ s.t.

- $\mathrm{label}(D_i) = x_i$,
- $\mathrm{frontier}(D_1).\mathrm{frontier}(D_2).\cdots.\mathrm{frontier}(D_l) = w'$.

Furthermore $\mathrm{size}([D_1, \ldots, D_l]) := \mathrm{size}(D_1) + \cdots + \mathrm{size}(D_l)$.

Lemma II.5.1.2. (Derivation Trees and Language Generation)

Lemma (II.5.1.2.)

Let $G = (T, N, S, P)$ be a CFG, $A \in T$, $w, w' \in (T \cup N)^*$, Then the following are equivalent

(1) *There exist a derivation forest D with root w and frontier w'.*

(2) $w \Rightarrow^* w'$.

In case $w' \in T^*$, the derivation sequence $w \Rightarrow^* w'$ can both be chosen as a left-most and as a right-most derivation sequence

Proof (1) ⇒ (2)

- The proof is by induction on $\mathrm{size}(D)$.
- We will do it in such a way that we get, in case all leaves are terminal symbols, a left-most derivation.
- A right most derivation can be obtained by choosing instead of the left-most the right most non-trivial derivation tree.
- Let $w = x_1 \cdots x_n$, $D = [D_1, \ldots, D_n]$.
- If all D_i are trivial, then $w' = w$, $w \Rightarrow^* w'$.
- So assume at least one D_i is non-trivial. Let D_k be the left-most non-trivial derivation tree, i.e. $D_1, \ldots, D_{k-1}$ are trivial, D_k is non-trivial.
- Let $D_k = \mathrm{tree}(A, D'_1, \ldots, D'_l)$, $\mathrm{label}(D'_i) = x'_i$. $x_k = A$.

Proof (1) ⇒ (2)

- Then
$$A \longrightarrow x'_1 \cdots x'_l$$
is a production, and we have that with
$$w_1 := x_1 \cdots x_{k-1} x'_1 x'_2 \cdots x'_l x_{k+1} x_{k+2} \cdots x_n$$
that
$$w = x_1 \cdots x_{k-1} A x_{k+1} \cdots x_n \Rightarrow w_1$$
is a one-step derivation.
In case $w' \in T^*$, we have $x_i \in T$, and this one-step derivation was left-most.
- We have that
$$[D_1, \ldots, D_{k-1}, D'_1, \ldots, D'_l, D_{k+1}, D_{k+2}, \ldots, D_n]$$
is a derivation forest with root w_1 and frontier w', and has size $\mathrm{size}(D) - 1$.

Proof (1) ⇒ (2)

- By IH there exist a derivation $w_1 \Rightarrow^* w'$, which in case of $w' \in T^*$ can be chosen as a left-most derivation sequence.
- Therefore $w \Rightarrow w_1 \Rightarrow^* w'$ is a derivation, which in case of $w' \in T^*$ can be chosen as a left-most derivation sequence.

(2) ⇒ (1)

- Proof is by induction on the length of the derivation $w \Rightarrow^* w'$.
- Let $w = x_1, \ldots, x_n$.
- In case the length is 0, $w' = w$, and we can choose $D = [\mathrm{tree}(x_1), \ldots, \mathrm{tree}(x_n)]$.
- Otherwise, assume that $x_k = A$ is a non-terminal, $A \longrightarrow y_1 \cdots y_l$ (with $y_i \in T \cup N$ or $l = 1 \wedge y_1 = \epsilon$). Let

$$w_1 := x_1 \cdots x_{k-1} A x_{k+1} \cdots x_n \Rightarrow x_1 \cdots x_{k-1} y_1 y_2 \cdots y_l x_{k+1} \cdots x_n$$

- Assume that the derivation is

$$w = x_1 \cdots x_{k-1} A x_{k+1} \cdots x_n \Rightarrow w_1 \Rightarrow^* w'$$

- By IH there exist a derivation forest

$$[D_1, \ldots, D_{k-1}, D'_1, \ldots, D'_l, D_{k+1}, \ldots, D_n]$$

with root w_1 and frontier w'.

(2) ⇒ (1)

- Now

$$[D_1, \ldots, D_{k-1}, (A, D'_1, \ldots, D'_l), D_{k+1}, \ldots, D_n]$$

is a forest with root w and frontier w'.

Lemma II.5.2.3. Uniqueness of Derivation (Trees)

Lemma (II.5.2.3.)

Let $G = (T, N, S, P)$ be a CFG, $A \in N$, $w \in T^*$.

(1) *Assume there are two different derivation trees with root labelled by A and frontier w. Then there exist two different left-most and two different right-most derivations of $A \Rightarrow^* w$.*

(2) *Assume there are two different left-most derivations or two different right-most-derivations of $A \Rightarrow^* w'$. Then there exist two different derivation trees of with root labelled by A and frontier w.*

Proof of Lemma II.5.2.3

- The example given in the general material demonstrates how derivation trees are transformed into left-most derivations in a unique way.
- A more formal proof is carried out by proving the following more general Lemma II.5.2.4:

Lemma II.5.2.4. Uniqueness of Derivation (Trees)

Lemma (II.5.2.4.)

Let $G = (T, N, S, P)$ be a CFG, $w \in (T \cup N)^+$, $w' \in T^*$.

(1) *Assume there are two different derivation forests with root w and frontier w'. Then there exist two different left-most and two different right-most derivations of $w \Rightarrow^* w'$.*

(2) *Assume there are two different left-most derivations or two different right-most-derivations of $w \Rightarrow^* w'$. Then there exist two different derivation forests of with root w and frontier w'.*

Proof of the Lemma II.5.2.4. (1)

- We prove only that there exist two different left-most derivations.
- Let $[D_1, \ldots, D_n]$ and $[D'_1, \ldots, D'_n]$ be two different derivation forests with root w and frontier w'.
- Induction on the length of the first derivation forest.
- If all D_i are trivial, then $w \in T^*$, $w = w'$, but then $D'_i = D_i$, which is not possible.
- Let D_k be the first non-trivial derivation tree, i.e. $D_1, \ldots, D_{k-1}$ are trivial.
- Let $D_k = \mathrm{tree}(A, D''_1, \ldots, D''_l)$, $\mathrm{label}(D''_i) = y_i$, $A \longrightarrow y_1 \cdots y_l$ a production.
- Let $w_1 := x_1 \cdots x_{k-1} y_1 \cdots y_l x_{k+1} \cdots x_n$.
- We have D'_i are trivial for $i < k$, D_k must be non-trivial (since it has as label a nonterminal).

Proof of the Lemma II.5.2.4. (1)

- Case 1: D'_k has the same production at the root, i.e. $D'_k = \mathrm{tree}(A, D'''_1, \ldots, D'''_l)$, $\mathrm{label}(D'''_i) = y_i$.
  - Then
  $$[D_1, \ldots, D_{k-1}, D''_1, \ldots, D''_l, D_{k+1}, \ldots, D_n]$$
  and
  $$[D'_1, \ldots, D'_{k-1}, D'''_1, \ldots, D'''_l, D'_{k+1}, \ldots, D'_n]$$
  must be different.
  - By IH there exist two different left-most productions $w_1 \Rightarrow w'$.
  - Therefore we obtain two different left-most productions $w \Rightarrow w_1 \Rightarrow w'$.

II.5.2. Uniqueness of Derivation Trees (14.1)

Proof of the Lemma II.5.2.4. (1)

- Case 2: D'_k has a different production at the root, i.e. $D'_k = \mathrm{tree}(A, D'''_1, \ldots, D'''_m)$, $\mathrm{label}(D'''_i) = y'_i$. $y'_1 \cdots y'_m \neq y_1 \cdots y_m$.
  - But then the first steps in the derivations constructed in the lemma are different, and we obtain two different left-most derivations.

II.5.2. Uniqueness of Derivation Trees (14.1)

Proof of the Lemma II.5.2.4. (2)

This proof is similar, at the first place where the two derivations differ we construct two different derivation forests.

II.5.2. Uniqueness of Derivation Trees (14.1)

Theorem II.5.2.4 Uniqueness of Derivation (Trees)

A direct consequence of the theorem is the following:

Theorem (II.5.2.4)

Let $G = (T, N, S, P)$ be a CFG, $w \in T^*$. The following are equivalent:

(1) *There exist exactly one derivation tree with label S and frontier w.*

(2) *There exist exactly one left-most derivation sequence $S \Rightarrow^* w$.*

(3) *There exist exactly one right-most derivation sequence $S \Rightarrow^* w$.*

Chomsky Normal Form

Definition

A CFG $G = (T, N, S, P)$ is in **Chomsky Normal From** if all of its productions are

$$\text{either of the form } A \longrightarrow BC \text{ or of the form } A \rightarrow a$$

where $A, B, C \in N$ and $a \in T$.

Chomsky Normal Form

Remark

If G is a CFG in Chomsky Normal Form, then $\epsilon \notin L(G)$.

Proof: Let $G = (T, N, S, P)$.
Since G has no production $A \longrightarrow \epsilon$, we have that if $w \Rightarrow w'$ then the $|w'| \geq |w|$.
Therefore if $w \Rightarrow^* w'$ then $|w'| \geq |w|$.
Therefore, if $w \in L(G)$ then $S \Rightarrow^* w$ therefore $|w| \geq |S| = 1$, $w \neq \epsilon$.

Chomsky Normal Form

We are going to prove the following:

Theorem

For any CFG G there exist a CFG G' in Chomsky Normal Form s.t.

$$L(G') = L(G) \setminus \{\epsilon\}$$

Step 1: Removal of null productions

Definition

A null-production of a grammar $G = (T, N, S, P)$ is a production of the form $A \longrightarrow \epsilon$.

Lemma

If G is a CFG. Then there exist a CFG G' which doesn't have any **null productions**, *and s.t. $L(G') = L(G) \setminus \{\epsilon\}$.*

Proof

Let $G = (T, N, S, P)$.
We first define the set of nullable nonterminals of G.
A nonterminal A is nullable if $A \longrightarrow \epsilon$.
They can be defined as follows:

- If $A \longrightarrow \epsilon$, then A is nullable.
- If $A \longrightarrow B_1 \cdots B_k$ for nullable nonterminals $B_1, \ldots, B_k$, then A is nullable.

Obtaining G' from G

We obtain the grammar $G' = (T, N, S, P')$ from $G = (T, N, S, P)$, by defining P' from P as follows:

- We start by taking all productions of P.
- We remove all productions $A \longrightarrow \epsilon$ from P.
- If $A \longrightarrow w$ is a production in P, and w' is obtained by omitting some but not all occurrences of nullable nonterminals from w, then

$$A \longrightarrow w'$$

  is added to P.

Equivalence of G and G'

- We show that $L(G') = L(G) \setminus \{\epsilon\}$.
- First $L(G') \subseteq L(G)$, because if $A \longrightarrow w'$ is a new production added to P', then $A \Rightarrow_G^* w'$, and therefore from any derivation in G' we obtain a derivation in G.

Equivalence of G and G'

- $L(G) \setminus \{\epsilon\} \subseteq L(G')$:
  - We show that from a derivation tree D of G with $\mathrm{frontier}(D) \neq \epsilon$ we obtain a derivation tree D' of G' with the same frontier and the same root by induction on the derivations.
  - If D is trivial, let $D' = D$.
  - Otherwise let $D = (A, D_1, \ldots, D_k)$.
  - Let $[D'_1, \ldots, D'_l]$ be obtained by
    - Omitting any D_i s.t. $\mathrm{frontier}(D_i) = \epsilon$.
    - Applying the IH to any D_i s.t. $\mathrm{frontier}(D_i) \neq \epsilon$ in order to obtain a derivation tree in G'.
  - Since $\mathrm{frontier}(D) \neq \epsilon$, the list $[D'_1, \ldots, D'_l]$ is not empty.
  - For the D_i omitted we have that $\mathrm{label}(D_i) \Rightarrow^*_G \epsilon$, so $\mathrm{label}(D_i)$ was nullable.
  - Therefore $\mathrm{label}(D'_1) \cdots \mathrm{label}(D'_l)$ is obtained from $\mathrm{label}(D_1) \cdots \mathrm{label}(D_k)$ by omitting some nullable nonterminals.
  - $A \longrightarrow \mathrm{label}(D_1) \cdots \mathrm{label}(D_k)$ was a production of G, and therefore $A \longrightarrow \mathrm{label}(D'_1) \cdots \mathrm{label}(D'_l)$ is a production of G'.

Equivalence of G and G'

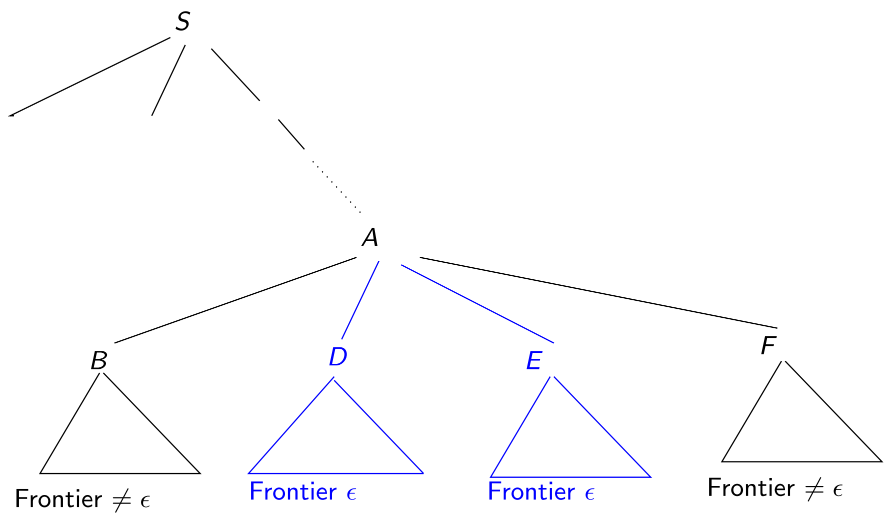
  - Therefore $D' := \mathrm{tree}(A, D'_1, \ldots, D'_l)$ is a derivation tree of G' and $\mathrm{frontier}(D) = \mathrm{frontier}(D')$.
- Now

$$\begin{array}{rcl} L(G) \setminus \{\epsilon\} & = & \{w \in T^* \setminus \epsilon \mid S \Rightarrow_G w\} \\ & \subseteq & \{w \in T^* \mid S \Rightarrow_{G'} w\} \\ & = & L(G') \end{array}$$

- The modification done can be described as follows:
- Assume a derivation tree D in G with root A and frontier $\neq \epsilon$.
- Take any subtree with frontier ϵ, which is not contained in a larger subtree with same frontier ϵ.
  - So any largest subtree with frontier ϵ.
- The root of subtree is nullable, since from it we derive ϵ.
- Omit this subtree with its root.
- The result of removing all such subtrees is a derivation tree in D'.

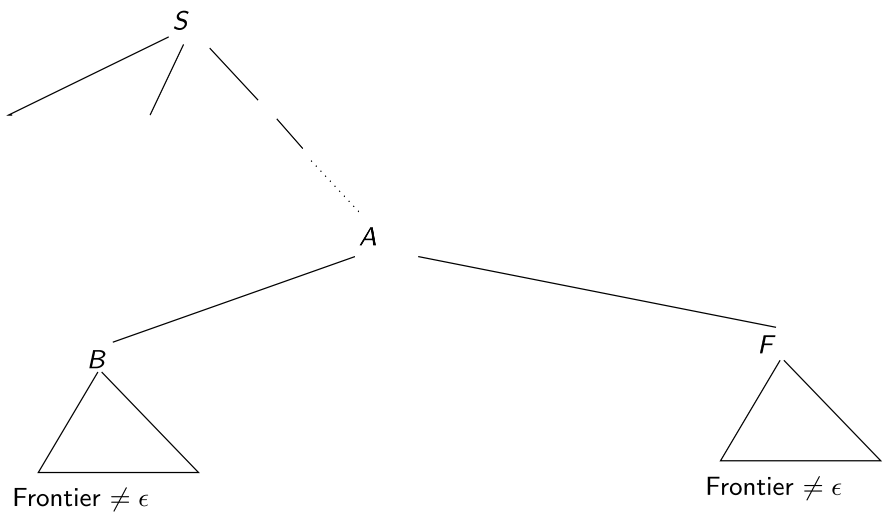
Picture

The blue subtrees will be removed:



Picture

After removing the subtrees:



Step 2: Removal of Silent Productions

Lemma

Assume G a CFG. Then there exist a CFG G' with no silent productions, i.e. no productions of the form $A \longrightarrow A'$ for nonterminals A, A' s.t.

$$L(G) = L(G')$$

If G has no null productions, G' can be chosen to have no null productions as well.

Proof

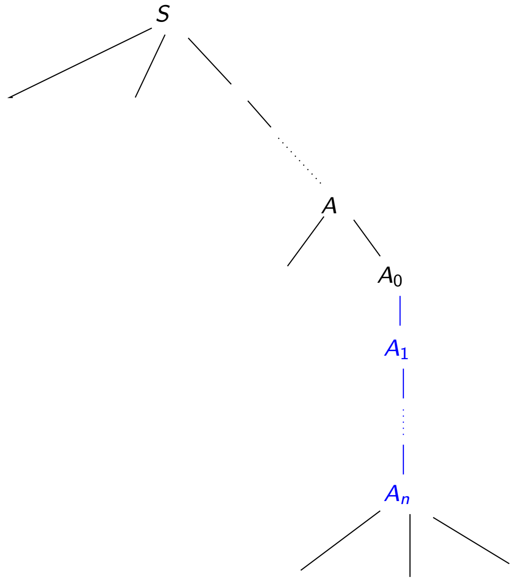
- Let $G = (T, N, S, P)$.
- Let $G = (T, N, S, P')$ where P' consists of all productions $A \longrightarrow w$ where $w \neq A$ and
$$A \Rightarrow^*_G B \longrightarrow w$$
- $L(G') \subseteq L(G)$, since any derivation step $uAv \Rightarrow_{G'} uwv$ where $A \longrightarrow_G w$ is as above can be replaced by $uAv \Rightarrow^*_G uBv \Rightarrow_G uwv$.

Proof of $L(G) \subseteq L(G')$

- We prove that if D is a derivation tree in G with frontier $w \in T^*$, then there exists a derivation tree D' in G' with the same root and frontier.
- Intuitively D' is obtained by contracting in D any chains of silent productions $A \longrightarrow A_1 \longrightarrow A_2 \longrightarrow \cdots \longrightarrow A_n$ where A_n is the head of a non-trivial derivation tree to A.

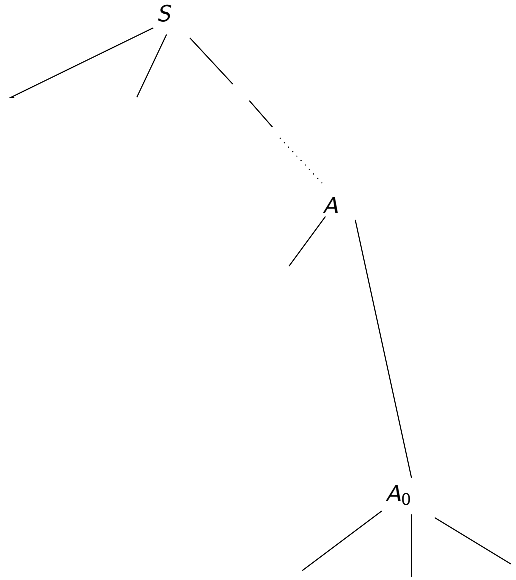
Picture

The blue chain of silent productions will be removed:



Picture

After removing the subtrees:



Removing Silent Productions

- Formally this can be done by induction on the derivation trees.
- Let D be a derivation tree with frontier w and root A.
- If $D = \mathrm{tree}(A)$, Let $D' = D$.
- If $D = \mathrm{tree}(A, D')$ with $\mathrm{label}(D') \in A$, let $D' = \mathrm{tree}(B, D_1, \ldots, D_n)$ be the derivation obtained from D' by IH.
  - Then we have
  $$A \Rightarrow_G B \Rightarrow_G^* C \longrightarrow \mathrm{label}(D_1) \cdots \mathrm{label}(D_k)$$
  so
  $$A \longrightarrow \mathrm{label}(D_1) \cdots \mathrm{label}(D_k)$$
  is a production of G'.
  - Therefore $\mathrm{tree}(A, D_1, \ldots, D_n)$ is a derivation as desired.

Removing Silent Productions

- Otherwise $D = \mathrm{tree}(A, D_1, \ldots, D_n)$ with $n \geq 2$ or $\mathrm{label}(D_1) \in T$.
  - Let D_i' a derivation of $\mathrm{frontier}(D_i)$ with $\mathrm{label}(D_i') = \mathrm{label}(D_i)$ obtained by the IH from D_i.
  - Then $\mathrm{tree}(A, D_1', \ldots, D_n')$ is a derivation as desired.

Step 3

Lemma

Let $G = (T, N, S, P)$ be a CFG. Then we can obtain a grammar G' s.t.

- *Productions in G' are either of the form $A \longrightarrow a$ for $A \in N$ and $a \in T$, or of the form $A \longrightarrow w$ where $w \in N^*$, so w consists only of nonterminals.*
- *$L(G') = L(G)$.*
- *If G has no null or silent productions, so does G'.*

Proof

Let $G = (T, N, S, P)$.

- Let $G' = (T, N', S, P')$ where
- N' is obtained by adding to N for each $a \in T$ a new symbol T_a.
- P' contains
  - Productions $T_a \longrightarrow a$ for $a \in T$,
  - For any production $T \longrightarrow w$ a production $T \longrightarrow w'$ where w' is obtained from w by replacing each terminal a by T_a.
- It is easy to see that $L(G') = L(G)$.

Proof of Chomsky Normal Form Theorem

- We prove now the Chomsky Normal Form Theorem.
- By step 1 - 3 there exists a grammar $G' = (T, N', S', P')$ s.t.
  - $L(G') = L(G) \setminus \{\epsilon\}$,
  - G' has no null or silent productions,
  - the productions of G' are either $A \longrightarrow a$ or $A \longrightarrow w$ with $w \in N^*$.
- Let $G'' = (T, N'', S', P'')$ where
  - P'' is obtained by replacing a production $A \longrightarrow B_1 B_2 \cdots B_n$ with $n \geq 3$ by

    $$A \longrightarrow B_1 C_1 \ , \ C_1 \longrightarrow B_2 C_2 \ , \ \ldots \ , \ C_{n-3} \longrightarrow B_{n-3} C_{n_2} \ , \\ C_{n-2} \longrightarrow B_{n-1} B_n$$

    for new symbols $C_1, \ldots, C_{n-2}$ (where for each production different symbols are added).
  - N'' is obtained from N by adding all new symbols C_i.

Proof of Chomsky Normal Form Theorem

- G'' is in Chomsky Normal Form.
- Obviously $L(G') \subseteq L(G'')$. since derivation steps

  $$vAw \Rightarrow vB_1 \cdots B_n w$$

  using production $A \longrightarrow B_1 \cdots B_n$ as above can be replaced by

  $$vAw \Rightarrow vB_1 C_1 w \Rightarrow vB_1 B_2 C_2 w \Rightarrow \cdots \Rightarrow vB_1 B_2 \cdots B_{n-3} C_{n-2} w \\ \Rightarrow vB_1 B_2 \cdots B_{n-3} B_{n-2} B_{n-1} w$$

Proof of Chomsky Normal Form Theorem

- On the otherhand, let for $w \in (T \cup N'')$ $\widehat{w}$ be defined as the result of replacing occurrences of
  - C_1 by $B_2 B_3 \cdots B_n$,
  - C_2 by $B_3 B_3 \cdots B_n$,
  - etc.
  - C_{n-2} by $B_{n-1} B_n$.
- Then for any production $D \longrightarrow_{G''} w$ of G'' we have either$D \longrightarrow_{G'} \widehat{w}$ or $D = \widehat{w}$.
- Therefore, if $S' \Rightarrow^*_{G''} w$, then $S' \Rightarrow^*_{G'} \widehat{w}$.
- Therefore using that for $w \in T^*$ we have $w = \widehat{w}$:

$$\begin{array}{rcl} L(G'') & = & \{w \in T^* \mid S' \Rightarrow^*_{G''} w\} \\ & = & \{\widehat{w} \mid w \in T^* \wedge S' \Rightarrow^*_{G''} w\} \\ & \subseteq & \{\widehat{w} \mid w \in T^* \wedge S' \Rightarrow^*_{G'} \widehat{w}\} \\ & = & \{w \in T^* \mid S' \Rightarrow^*_{G'} w\} \\ & = & L(G') \end{array}$$

A recognition Algorithm for CFG (14.3.2)

- We present an algorithm for deciding for a CFG G in Chomsky Normal Form and a string w whether $w \in L(G)$.
- This algorithm is called the **Cocke-Younger-Kasami (CYK) algorithm**.

Cocke-Younger-Kasami Algorithm

- Let $G = (T, N, S, P)$.
- For a word $w = t_1 \cdots t_n$ let for $1 \leq i \leq j \leq n$ $w_{i,k}$ be the subword starting from t_i of length k, i.e.

$$w_{i,k} = t_i t_{i+1} \cdots t_{i+k-1}$$

- We decide more generally for any $1 \leq i \leq n$, $1 \leq k \leq n - i + 1$ and $A \in N$ whether $T \Rightarrow^* w_{i,j}$.
- Let

$$N_{i,j} := \{A \in N \mid A \Rightarrow^* w_{i,j}\}$$

Cocke-Younger-Kasami Algorithm

- We have
  - $N_{i,1} = \{A \in N \mid A \longrightarrow w_{i,1}\}$, since a derivation $A \Rightarrow w_{i,1}$ cannot start with $A \Rightarrow BC$ since otherwise we would have $1 = |w_{i,1}| \geq |BC| = 2$.
  - We have

$$\begin{array}{rl} N_{i,j+1} = \{A \mid \exists A \longrightarrow BC \in P.\exists k < j + 1.1 \leq k \wedge \\ B \in N_{i,k} \wedge C \in N_{i+k,j+1-k}\} \end{array}$$

  A derivation tree of $w_{i,j}$ must start with $A \longrightarrow BC$. Then B, C derive two subwords of $w_{i,j}$ which together form $w_{i,j}$, both of which are non-empty, so $B \in N_{i,k}$, $C \in N_{i+k,j+1-k}$ for some $1 \leq k \leq j$ as above.

Cocke-Younger-Kasami Algorithm

- So we can define $N_{i,j}$ by the following algorithm:

for $i := 1$ **to** n **do begin**
  $N_{i,1} := \{A \mid A \longrightarrow w_{i,1} \in P;\}$
  end
for $j := 2$ **to** n **do begin**
  for $i := 1$ **to** $n + 1 - j$ **do begin**
    $N_{i,j} := \emptyset;$
    for $k := 1$ **to** $j - 1$ **do begin**
      $N_{i,j} := N_{i,j} \cup \{A \mid \exists B, C.A \longrightarrow BC \in P$
      $\wedge B \in N_{i,k} \wedge C \in N_{i+k,j-k}\}$
      end
    end
  end

Cocke-Younger-Kasami Algorithm

- Now we have for $w \in T^*$

$$\begin{aligned} w \in L(G) &\Leftrightarrow S \Rightarrow^* w \\ &\Leftrightarrow S \in N_{1,|w|} \end{aligned}$$

- The above algorithm runs in cubic time in $|w|$.

Pumping Lemma for CFG (Repetition from Main Slides)

Theorem

Let L be a context free language.. Then there exists a constant k s.t. for all strings z of L s.t. $|z| \geq k$ there exist u, v, w, x, y s.t.

- $z = uvwxy$,
- $|vwx| \leq k$, *i.e. the middle portion is not too long,*
- $|vx| \geq 1$, *i.e. v or x are not ϵ,*
- $\forall i \geq 0.uv^iwx^iy \in L$.

Proof of the Pumping Lemma for CFG

- For simplicity we consider CFGs in Chomsky-Normal-Form.

Lemma (Height of Derivation)

In order to prove the pumping lemma, we need to prove some lemmas.

The first one relates the height of a derivation to the length of the derived string:

Lemma

Let $G = (N, V, S, P)$ be a CFG in Chomsky Normal Form, D a derivation tree of $w \in T^*$ of height $n \geq 1$. Then $|w| \leq 2^{n-2}$.

Proof of Lemma (Height of Derivation)

Proof by induction on n.

- If $n = 2$, then $D = \mathrm{tree}(A, \mathrm{tree}(w))$, $w \in T$. $|w| = 1 = 2^{n-2}$.
- $2 \leq n \to n+1$: The rule used at the root must have been $A \to BC$, since, if we had used $A \to a$, we would get a tree of height 2. Let $D = \mathrm{tree}(A, D_1, D_2)$. Then

$$
\begin{aligned}
|w| &= |\mathrm{frontier}(w_1).\mathrm{frontier}(w_2)| \\
&= |\mathrm{frontier}(w_1)| + |\mathrm{frontier}(w_2)| \\
&\leq 2^{n-2} + 2^{n-2} = 2^{n+1-2}
\end{aligned}
$$

Definition of Subderivations

Definition

Let $G = (T, N, S, P)$ be a grammar, D', D derivations in G.
We define what it means for D' to be a **subderivation of D**:

- D is a subderviation of itself.
- If D'' is a child of D, and D' is a subderivation of D' then D' is a subderivation of D.

D' is a **proper subderivation of D** if D' is a subderivation of D, but $D' \neq D$.

Existence of Subderivations of smaller heights

Lemma

Let D be a derivation of height n, $1 \leq k < n$. Then there exist a proper subderivation D' of D of height k.

Proof of Lemma (Existence of Subderivations)

- The proof is by induction on the height n of D:
- In case $n = 1$, no $1 \leq k < n$ exists.
- In case $n = 2$, $D = \mathrm{tree}(A, D_1, \ldots, D_l)$. We must have $l \geq 1$, otherwise $\mathrm{height}(D) = 1$. $\mathrm{height}(D_1) = 1 = k$, let $D' = D_1$.
- Induction step $n \to n + 1$, assuming $n \geq 2$.
  - Let $D = \mathrm{tree}(A, D_1, \ldots, D_l)$.
  - $n + 1 = \mathrm{height}(D) = \max\{\mathrm{height}(D_1), \ldots, \mathrm{height}(D_l)\} + 1$,
  - So there exists an i s.t. $\mathrm{height}(D_i) = n$.
  - If $k = n$, choose $D' := D_i$.
  - If $k < n$, then by IH there exist a proper subderivation D'' of D_i of height k, which is a proper subderivation of D as well. $D' := D''$.

Derivations Deriving Non-Empty Strings

Lemma

Assume $G = (T, N, S, P)$ is a CFG with no null-productions. Then if $A \Rightarrow^* w$, then $|w| \geq 1$.

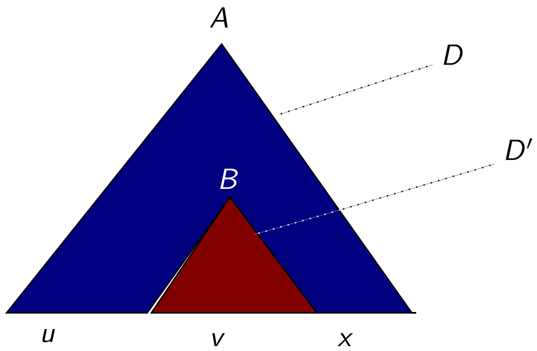
Proof: If $v \Rightarrow v'$, then $|v'| \geq |v|$. Therefore if $v \Rightarrow^* v'$, then $|v'| \geq |v|$. Now $|w| \geq |A| = 1$.

Cutting out Subderivations

Lemma

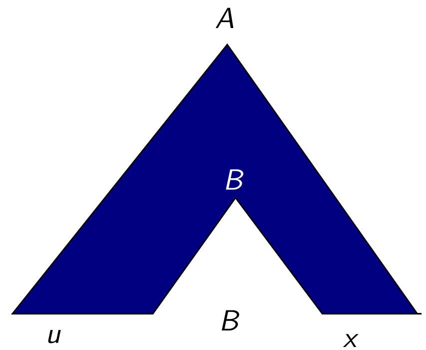
Let G be a CFG.
Let D be a derivation tree with label A and frontier w. Let D' be a proper subderivation D' of D with label B and frontier v.
Then there exist u, x s.t. $w = uvx$, and a derivation $A \Rightarrow^* uBx$.
Furthermore, if $\mathrm{height}(D) \geq 2$ and D has no null or silent productions, then $|u| + |x| \geq 1$.

Picture



Picture

After cutting out the derivation:



Proof (Cutting out Subderivations)

Induction on D' being a subderivation of D.
We write "the special case" for the condition
"$\mathrm{height}(D) \geq 2$ and D has no null or silent productions".
Let $D = \mathrm{tree}(A, D_1, \ldots, D_n)$.
In the special case we get $n \geq 2$.

- (If $n = 1$, we would obtain $D_1 = \mathrm{tree}(a, D'_1, \ldots, D'_l)$, a must be a nonterminal, since G has no silent productions, and therefore $l = 0$. But then $\mathrm{height}(D) = 1$).

Proof (Cutting out Subderivations)

Case 1: D' is a child of D.
Then $D = \mathrm{tree}(A, D_1, \ldots, D_n)$, $D' = D_i$ some i.
Let

$$\begin{array}{rcl} u & := & \mathrm{frontier}(D_1).\mathrm{frontier}(D_2).\cdots.\mathrm{frontier}(D_{i-1}) \\ x & := & \mathrm{frontier}(D_{i+1}).\mathrm{frontier}(D_{i+2}).\cdots.\mathrm{frontier}(D_n\ . \end{array}$$

Then $w = uvx$. Let

$$D'' := \mathrm{tree}(A, D_1, \ldots, D_{i-1}, \mathrm{tree}(B), D_{i+1}, \ldots, D_n) \ .$$

D'' is a derivation tree of uBx with the label A.
Furthermore, in the special case we have $n \geq 2$, $\mathrm{frontier}(D_i) \neq \epsilon$, therefore $u \neq \epsilon$ or $x \neq \epsilon$.

Proof (Cutting out Subderivations)

Case 1: D' is a proper subderivation of a child D_i of D. Let C be the label of D_i.
Let $w' := \mathrm{frontier}(D_i)$. By IH there exists u', x' s.t. $w' = u'vx'$, and a derivation tree D' with label C and frontier $u'Bx'$.

$$\begin{array}{rcl} u'' & := & \mathrm{frontier}(D_1).\mathrm{frontier}(D_2).\cdots.\mathrm{frontier}(D_{i-1}) \\ x'' & := & \mathrm{frontier}(D_{i+1}).\mathrm{frontier}(D_{i+2}).\cdots.\mathrm{frontier}(D_n) \ . \end{array}$$

Then $w = u''w'x'' = (u''u')v(x'x'')$. Let $u := u''u'$, $x := x'x''$. Let

$$D'' := \mathrm{tree}(A, D_1, \ldots, D_{i-1}, D', D_{i+1}, \ldots, D_n) \ .$$

D'' is a derivation tree of $u''u'Bx'x'' = uBx$.
Furthermore, in the special case we have as before $u'' \neq \epsilon$ or $x'' \neq \epsilon$, therefore $u \neq \epsilon$ or $x \neq \epsilon$.

Chains of Derivations

Definition

Let G be a CFG. A **chain of derivations** is a list of derivations $[D_1, \ldots, D_n]$ s.t. D_{i+1} is a child of D_i and $\mathrm{height}(D_n) = 1$. n is called the **length** of the chain of derivations.

Existence of Chains of Derivations

Lemma

Let G be a CFG. If D has height n then there exists a chain of derivations $[D_1, \ldots, D_n]$ of length n s.t. $D_1 = D$.

Proof (Existence of Chains of Derivations)

Induction on n: If $n = 1$, then $[D]$ is a chain of derivations as desired.
$n \to n+1$: Let D have height $n+1$.
There exist a child D_2 of $D_1 := D$ which has height n.
By IH there exist a chain of derivations $[D_2, D_3, \ldots, D_{n+1}]$ of length n starting with D_2.
Then $[D_1, \ldots, D_{n+1}]$ is a chain of derivations as desired.

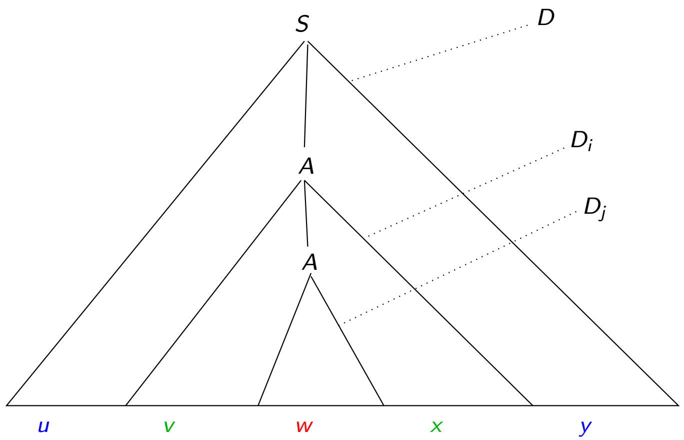
Proof of the Pumping Lemma

- Let $L = L(G)$. W.l.o.g. G is in Chomsky Normal Form.
  - There exist a grammar G' in Chomsky Normal Form s.t. $L(G') = L \setminus \{\epsilon\}$.
  - If the lemma holds for $L(G')$, then it holds for L as well (with the same constant n).
- Let $G = (T, N, S, P)$, and assume $|N| = l$.
- Let $k := 2^l$.

Proof of the Pumping Lemma

- Assume $z \in L(G)$, $|z| \geq k = 2^l$.
- Let D be a derivation tree for z.
- $\mathrm{height}(D) \geq l + 2$.
- There exist a subderivation D' of D of height $l + 2$.
- Then there exist a chain of derivations $[D_1, D_2, \ldots, D_{l+2}]$, s.t. $D' = D_1$.
- $\mathrm{label}(D_i) \in N$ for $i = 1, \ldots, l + 1$.
- Therefore there exists $1 \leq i < j \leq l + 1$ s.t. $\mathrm{label}(D_i) = \mathrm{label}(D_j) =: A$.
- So we have found a subderivation D_i of D of height $\leq l + 2$ which contains a proper subderivation D_j with the same label.

Picture

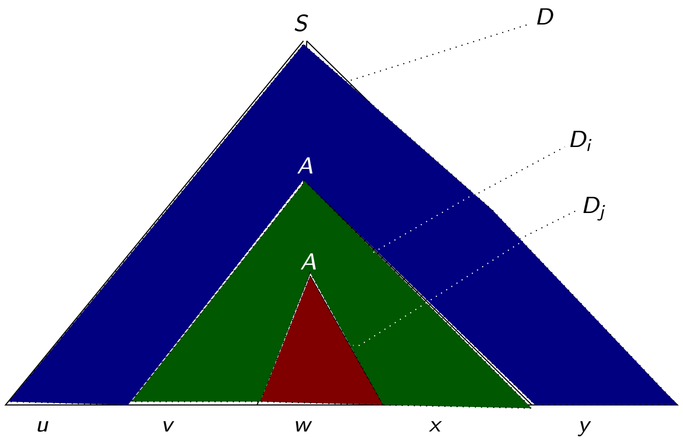


Proof of the Pumping Lemma

- Let $w = \mathrm{frontier}(D_j)$.
- By the "Cutting out of Derivations" lemma applied to D_i and D_j, we have $\mathrm{frontier}(D_i) = vwx$ s.t. $vx \neq \epsilon$, and $A \Rightarrow^* vAx$. (The green derivation on the next picture).
- By the "Cutting out of Derivations" lemma applied to D and D_i we have $z = uvwxy$ for some strings u, y, and we have $S \Rightarrow^* uAy$. (The blue derivation on the next picture).
- Furthermore by D_j we have $A \Rightarrow^* w$. (The red derivation on the next picture).
- $\mathrm{height}(D_i) \leq l + 2$, therefore $|vwx| \leq 2^l = k$.
- We obtain

$$
\begin{array}{l}
A \Rightarrow^* vAx \Rightarrow^* v^2Ax^2 \Rightarrow^* \cdots \\
\forall i \geq 0. A \Rightarrow^* v^iAx^i \\
\forall i \geq 0. S \Rightarrow^* uAy \Rightarrow^* uv^iAx^iy \Rightarrow^* uv^iwx^iy \\
\forall i \geq 0. uv^iwx^iy \in L(G)
\end{array}
$$

Picture



Floyd's Theorem

- We are going to show Floyd's Theorem, that a programming language with variable declarations cannot be defined in a context free way.
- We need some restrictions on what is meant by a programming language.
- In order to illustrate it, be consider a restricted form of the while language already considered.

Grammar for While Programs

Consider the grammar studied in 2.1.3:

grammar	G^{while}
import	$G^{Arithmetic\_Expression}$, $G^{Boolean\_Expression}$
terminals	**skip**, **if**, **then**, **else**, **fi**, **while**, **do**, **od**, :=, ;
nonterminals	*Program*
start symbol	*Program*
productions	*Program*⟶**skip** *Program*⟶*Id* := *AExp* *Program*⟶*Program* ; *Program* *Program*⟶**if** *BExp* **then** *Program* **else** *Program* **fi** *Program*⟶**while** *BExp* **do** *Program* **od**

Modified Grammar

We simplify it by adding variable declarations, and omit **if_then_else** and **while** and disambiguate the sequencing construct. We obtain the following grammar:

Grammar for Programs With Declarations

grammar	$G^{Programs\_with\_declarations}$
import	$G^{Arithmetic\_Expression}$, $G^{Declarations}$
terminals	**skip**, **begin**, **end**, :=, ;
nonterminals	*Program*, *CommandList*, *Command*
start symbol	*Program*
productions	*Program* ⟶ **begin** *Declaration* ; *CommandList* **end**
	CommandList ⟶ *Command*
	CommandList ⟶ *CommandList* ; *Command*
	Command ⟶ **skip**
	Command ⟶ *Id* := *AExp*

Declarations

grammar	$G^{Declarations}$
import	$G^{Identifier}$
terminals	**nat**
nonterminals	*Declaration*, *IdentifierList*
start symbol	*Declaration*
productions	*Declaration* ⟶ **nat** *IdentifierList*
	IdentifierList ⟶ *Identifier*
	IdentifierList ⟶ *Identifier* , *IdentifierList*

Floyd's Theorem II.5.4

Theorem

Let

$$Decl_{n,m} := \textbf{begin nat } a^n b^m \; ; \; a^n b^m := 0 \textbf{ end}$$

Let

$$Decl_{n,m} \subseteq L$$

be a programming language.
Assume some sanity condition on what it means for L to be a programming langauge, including that in L all identifiers need to be declared before being used.
Then L is not context free.

Floyd's Theorem II.5.4

Theorem (Cont)

The sanity conditions are the following:

1. *In any program of L all identifiers of the program are declared by strings t a where t is a type identifier, which are special elements of the alphabet.*
2. **nat** *is one type identifier.*
3. *The keywords and ":=" are special elements of the alphabet.*
4. *Identifiers are strings.*
5. **begin** *and* **end** *need to be balanced in L.*
6. *For an identifier s* $s := 0$ *is an instruction using identifier s.*
7. *Removing of one or more of the symbols* $:=$, 0 *from* $s := 0$ *yields a string which is not an instruction.*
8. *Programs in L contain at least one instruction.*

Proof (Floyd's Theorem)

- Assume L is context free and let k be the constant of the pumping lemma for CFG.
- Let $n, m > k$.
- $Decl_{n,m} \in L$ and $|Decl_{n,m}| > k$.
- By the pumping lemma there exists u, v, w, x, y s.t.

$$Decl_{n,m} = uvwxy \wedge vx \neq \epsilon \wedge |vwx| \leq k \wedge \forall i \geq 0. uv^i wx^i y \in L$$

- Let $P_i := uv^i wx^i y$.
- So

$$uvwxy = Decl_{n,m} = \textbf{begin nat } a^n b^m \text{ ; } a^n b^m := 0 \textbf{ end}$$

Proof (Floyd's Theorem)

$uvwxy = Decll_{n,n} = \textbf{begin nat } a^n b^m \text{ ; } a^n b^m := 0 \textbf{ end}$
$P_i := uv^i wx^i y \in L$

- Because $|vwx| \leq k$, vwx cannot containing both **begin** and **end**.
- If v or x contained **begin**, or **end**, then P_0 would contain only one of these two keywords, contradicting the fact that **begin** and **end** need to be balanced.
- If v contains **nat**, then vwx must be part of **nat** $a^n b^n$.
  Then P_0 wouldn't contain a declaration of $a^n b^m$ which is used in the statement $a^n b^m := 0$ which is part of P_0.
- Therefore we have that vwx must be part of $a^n b^m$; $a^n b^m := 0$.

Proof (Floyd's Theorem)

$uvwxy = Decll_{n,n} = \textbf{begin nat } a^n b^m \text{ ; } a^n b^m := 0 \textbf{ end}$
$P_i := uv^i wx^i y \in L$
vwx part of $a^n b^m$; $a^n b^m := 0$

- If $:= 0$ would overlap with x, then P_0 would not have one or more of those symbols, and the part after ; wouldn't be an instruction.
- So vwx is part of $a^n b^m$; $a^n b^m$.

Proof (Floyd's Theorem)

$uvwxy = Decl l_{n,n} =$ **begin nat** $a^n b^m$; $a^n b^m := 0$ **end**
$P_i := uv^i wx^i y \in L$
vwx part of $a^n b^m$; $a^n b^m$

- If v or x contain ; then P_0 doesn't contain a ;, so it consists only of a declaration and has no instruction.
- If w contains ;, then v is contained in b^m, x contained in a^n, therefore in P_0 the declared identifier and used identifier are different, contradicting that identifiers need to be declared.
- So vwx is part of $a^n b^m$ before or after the ;. But then in P_0 the identifiers before and after ; are different, contradicting that identifiers need to be declared.