II.2.1. Regular Languages (12.2)

Finite Languages are Regular

<table>
<thead>
<tr>
<th>grammar</th>
<th>$G_{ab,aabb,aaabbb}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>terminals</td>
<td>$a, b$</td>
</tr>
<tr>
<td>nonterminals</td>
<td>$S$</td>
</tr>
<tr>
<td>start symbol</td>
<td>$S$</td>
</tr>
<tr>
<td>productions</td>
<td>$S \rightarrow ab$</td>
</tr>
<tr>
<td></td>
<td>$S \rightarrow aabb$</td>
</tr>
<tr>
<td></td>
<td>$S \rightarrow aaabbb$</td>
</tr>
</tbody>
</table>

This grammar is not regular, since there can only be one terminal in the right hand string. But we can amend this:
Finite Languages are Regular

Lemma II.2.1.1.

The following left-linear grammar generates \( \{a^m b^n \mid m, n \geq 1\} \).

<table>
<thead>
<tr>
<th>Grammar</th>
<th>( G^{\text{left-linear}, a^m b^n} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Terminals</td>
<td>( a, b )</td>
</tr>
<tr>
<td>Nonterminals</td>
<td>( S, T )</td>
</tr>
<tr>
<td>Start Symbol</td>
<td>( S )</td>
</tr>
</tbody>
</table>
| Productions | \( S \rightarrow Sb \)  
               | \( S \rightarrow Tb \)  
               | \( T \rightarrow Ta \)  
               | \( T \rightarrow a \)  |
A Right-Linear Grammar for $a^m b^n$

The following right-linear grammar generates $\{a^m b^n \mid m, n \geq 1\}$:

```
grammar $G_{\text{right-linear}, a^m b^n}$
terminals $a, b$
nonterminals $S, T$
start symbol $S$
productions $S \rightarrow aS$
   $S \rightarrow aT$
   $T \rightarrow bT$
   $T \rightarrow b$
```

Right-Linear Grammar for Numbers

Here is a right-linear grammars for numbers without leading zeros. We use “|” as for BNF.

```
grammar $G_{\text{Number}}$
terminals 0, 1, ..., 9
nonterminals Number, Digits
start symbol Number
productions Number $\rightarrow$ 0
   Number $\rightarrow$ 1 Digits $|$ 2 Digits $|$ ... $|$ 9 Digits
   Digits $\rightarrow$ 0 Digits $|$ 1 Digits $|$ ... $|$ 9 Digits
   Digits $\rightarrow$ $\epsilon$
```

Why didn’t we use the following as in BNF?

```
grammar $G_{\text{Number}}$
terminals 0, 1, ..., 9
nonterminals Number, Digit, NonZeroDigit, Digits
start symbol Number
productions Number $\rightarrow$ Digit $|$ NonZeroDigit Digits
   Digits $\rightarrow$ Digit $|$ Digit Digits
   Digit $\rightarrow$ 0 $|$ NonZeroDigit
   NonZeroDigit $\rightarrow$ 1 $|$ 2 $|$ ... $|$ 9
```

Answer:

Right-Linear Grammar for Post Codes

The next grammar generates the postcodes of the form SA1 8PP or in general LLd dLL for digits d and capital letters L without any leading zeros. We use the notation | as in BNF. We write $\text{[blank]}$ for blank.

```
grammar $G_{\text{Number}}$
terminals 0, 1, ..., 9
nonterminals Number, Digits
start symbol Number
productions Number $\rightarrow$ 0
   Number $\rightarrow$ 1 Digits $|$ 2 Digits $|$ ... $|$ 9 Digits
   Digits $\rightarrow$ 0 Digits $|$ 1 Digits $|$ ... $|$ 9 Digits
   Digits $\rightarrow$ $\epsilon$
```

Answer:
II.2.1. Regular Languages (12.2)

Right-Linear Grammar for Post Codes

<table>
<thead>
<tr>
<th>grammar</th>
<th>( G^{Postcode} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>terminals</td>
<td>0, 1, \ldots, 9, A, B, \ldots, Z, ⌞⌟</td>
</tr>
<tr>
<td>nonterminals</td>
<td>postcode, letter2, digit1, blank1, digit2, letter3, letter4</td>
</tr>
<tr>
<td>start symbol</td>
<td>postcode</td>
</tr>
<tr>
<td>productions</td>
<td>postcode ( \rightarrow A ) letter2</td>
</tr>
<tr>
<td></td>
<td>letter2 ( \rightarrow A ) digit1</td>
</tr>
<tr>
<td></td>
<td>digit1 ( \rightarrow 0 ) blank1</td>
</tr>
<tr>
<td></td>
<td>blank1 ( \rightarrow ⌞⌟ ) digit2</td>
</tr>
<tr>
<td></td>
<td>digit2 ( \rightarrow 0 ) letter3</td>
</tr>
<tr>
<td></td>
<td>letter3 ( \rightarrow A ) letter4</td>
</tr>
<tr>
<td></td>
<td>letter4 ( \rightarrow A )</td>
</tr>
</tbody>
</table>

Example Derivation

Here is a derivation of \( SA2_{\text{PP}}8PP \in L(G^{Postcode}) \):

\[
\begin{align*}
\text{postcode} & \Rightarrow S \text{ letter2} \\
& \Rightarrow SA \text{ digit1} \\
& \Rightarrow SA1 \text{ blank1} \\
& \Rightarrow SA1_{\text{PP}} \text{ digit2} \\
& \Rightarrow SA1_{\text{PP}}8 \text{ letter3} \\
& \Rightarrow SA1_{\text{PP}}8P \text{ letter4} \\
& \Rightarrow SA1_{\text{PP}}8PP
\end{align*}
\]

Easier Proof that Postcodes are Regular

Can you give an easier proof that the language of postcodes is regular (both left-linear and right-linear)?

Multi-step Regular Grammars

- In general we can extend regular grammars by allowing productions such as

  \[
  \begin{align*}
  S & \rightarrow abB \\
  B & \rightarrow aS \\
  B & \rightarrow baS
  \end{align*}
  \]

  So instead of having only one terminal symbol, we can have several.

- As long as we remain left-linear or right-linear
  - i.e. the terminal symbols are always to the right or always to the left of the non-terminal on the right hand side of a rule

  we obtain grammars which can be reduced to regular grammars.
Lemma II.2.1.2.

Lemma (II.2.1.2.)

1. Assume a grammar $G$ which has only productions of the form
   \[ A \rightarrow Bw \text{ or } A \rightarrow w \]
   for some $w \in T^*$, $A, B \in N$. Then $L(G) = L(G')$ for some left-linear grammar $G'$, which can be computed from $G$.

2. Assume a grammar $G$ which has only productions of the form
   \[ A \rightarrow wB \text{ or } A \rightarrow w \]
   for some $w \in T^*$, $A, B \in N$. Then $L(G) = L(G')$ for some right-linear grammar $G'$, which can be computed from $G$.

Proof Idea for Lemma II.2.1.2.

- First omit all so called silent productions, i.e. productions of the form $A \rightarrow B$ for some non-terminals $A, B$.
  - This requires some work.
- Then replace in the right-linear case productions
  \[ A \rightarrow a_1 a_2 \cdots a_n B \]
  with $n \geq 2$ by productions
  \[ A \rightarrow a_1 A_1, \]
  \[ A_1 \rightarrow a_2 A_2, \]
  \[ \ldots \]
  \[ A_{n-1} \rightarrow a_n B \]
  for some new non-terminals $A_i$.
- Full details can be found in the additional material.

Remark

In a regular grammar we are not allowed to mix left-linear and right-linear grammars. Otherwise we would obtain truly context-free languages.

We call grammars as above multi-step right-linear/left-linear/regular grammars.
Example (Mixing Left/Right-Linear Rules)

The following grammar generates the language $L(G) = \ ?$
which (as we will see later) is context-free but not regular.

<table>
<thead>
<tr>
<th>Grammar</th>
<th>$G$</th>
</tr>
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<tbody>
<tr>
<td>Terminals</td>
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<tr>
<td>Nonterminals</td>
<td>$S, T$</td>
</tr>
<tr>
<td>Start Symbol</td>
<td>$S$</td>
</tr>
</tbody>
</table>
| Productions | $S \rightarrow ab$
$S \rightarrow aT$
$T \rightarrow Sb$ |

Operators for Forming Languages

Definition

Let $L_1, L_2, L \subseteq T^*$ be languages over the alphabet $T$.

1. The **concatenation** $L_1 . L_2$ of $L_1$ and $L_2$ is defined as

   $$L_1 . L_2 := \{ w_1 w_2 \mid w_1 \in L_1, w_2 \in L_2 \}$$

2. The **union** $L_1 \mid L_2$ of $L_1$ and $L_2$ is defined as

   $$L_1 \mid L_2 := L_1 \cup L_2$$

   The union is sometimes denoted by $+$. 

3. The **iteration** or **Kleene-star** $L^*$ of $L$ is defined as

   $$L^* := \{ w_1 w_2 \cdots w_n \mid n \geq 0, w_1, \ldots, w_n \in L \}$$

Regular expressions are denotations for languages formed from the $\emptyset$ and finite sets using the above mentioned operations.
Regular Expressions

Definition
Let \( T \) be an alphabet. We define the set of \textit{regular expressions} over an alphabet \( T \) inductively together with the language \( L(E) \) for each regular expression \( E \).

- \( \emptyset \) is a regular expression, \( L(\emptyset) := \emptyset \).
- \( \epsilon \) is a regular expression, \( L(\epsilon) := \{\epsilon\} \).
- For \( a \in T \) we have \( a \) is a regular expression, \( L(a) := \{a\} \). One usually writes \( a \) for the regular expression, when the symbol is \( a \).
- If \( E, F \) are regular expressions, then
  - \( (E) \mid (F) \) is a regular expression, \( L((E) \mid (F)) := L(E) \cup L(F) \).
  - \( (E)(F) \) is a regular expression, \( L((E)(F)) = L(E).L(F) \).
  - \( (E)^* \) is a regular expression, \( L((E)^*) = L(E)^* \).

We omit unnecessary brackets and usually write \( E \mid F \) instead of \( (E) \mid (F) \), \( EF \) instead of \( (E)(F) \), \( E^* \) instead of \( (E)^* \), if there is no confusion.

Use of Regular Expressions

- We will usually omit writing \( L(E) \), so write \( (0 1) 0^* \)
  instead of \( L((0 1) 0^*) \)
  which is \( \{\{0\}.\{1\}\}.\{\{0\}\}^* \)
- We will as well identify regular expressions which denote the same language. Therefore we can omit more brackets e.g. we can write
  \[ 0 1 0 \]
  instead of
  \[ (0 1) 0 \]

Examples of Regular Expressions

- If the alphabet only contains single characters, we can omit the blank in concatenation, and write
  \[ 010 \] instead of \( 0 1 0 \)
- The set of non-zero digits is defined as
  \[ \text{NonzeroDigit} = 1 \mid 2 \mid \cdots \mid 9 \]
- The set of digits is defined as
  \[ \text{Digit} = 0 \mid \text{NonZeroDigit} \]
- The set of numbers without leading zero is
  \[ \text{Number} = 0 \mid (\text{NonZeroDigit} \text{Digit}^*) \]
- The set of capital letters is defined by
  \[ \text{CapitalLetter} = A \mid B \mid \cdots \mid Z \]
Examples of Regular Expressions

- The set of module codes in this department is
  
  \[ CS_{\text{ModuleCodes}} = \text{CS} - (0 | 1 | 2 | 3 | M) \text{ Digit Digit} \]

Examples of Regular Expressions

- The set of postcodes can be defined as
  
  \[ \text{postcode} = \text{CapitalLetter CapitalLetter Digit } [\text{Digit CapitalLetter CapitalLetter} \]
Notations for Regular Expressions

- One writes \([a_1 \cdots a_n]\) for \(a_1 | \cdots | a_n\).
- One writes \([a-z]\) for \([a, b, c, \ldots z]\) similarly for \([0-9], [A-Z]\).
- \([a-zA-Z0-9]\) := \([a-z][A-Z], [a-z]?: \) etc.
- One writes \(L^+\) or \(L^+\) for \(L L^+\) (so \(L^+ := \{w_1 \cdots w_n | n \geq 1, w_1, \ldots, w_n \in L\}\), the set of words formed from \(L\) by using at least one word in \(L\).

Question: Is \(L^+\) the set of non-empty words formed from elements of \(L\)?
Answer:

- Lots of other useful operators for constructing regular expressions have been defined.
- Each language has its own set and of regular expressions (using often different notations), and its own syntax. Sometimes operators are introduced which go beyond regular languages.

Example Use of Regular Expressions

Assume you have files called automata\text{theorych1}.\text{tex}, automata\text{theorych2}.\text{tex}, automata\text{theorych3}.\text{tex}, \ldots
Concatenation all of them into one file:

```
cssetzer@cs-svr1:\> cat automata\text{theory}[0-9].\text{tex} >
automata\text{theoryall}.\text{tex}
```

Process lines in a file containing entries separated by “,”, do something if the first field is a student number (a string consisting of digits only). Python code

```python
define(filename):
define = re.compile(r'^[0-9]*$')
    for line in define:
        a = line.split(',
        if define.match(a[0]):
            print a[1][:-1] #cut off trailing \\
        define.close()
```

Example WebLinks

Consider links in http pages of the form:

```
\<a href="http://www.swan.ac.uk/"\>Swansea University\</a\>
```

displayed as

Swansea University

The set of weblinks can be defined as
(in most language \(\_\_\) would be written as a blank, \(\"\) would be preceded by a \(\backslash\), and the whole string would be put into quotation marks):

```
webLinks = "\<a href="[a-zA-Z0-9/\_\_]*\"">[a-zA-Z0-9/\_\_]</a>"
```

E.g. in Python one would write

```
weblinks=\"<a href=\"[a-zA-Z0-9/\_\_]\">[a-zA-Z0-9 \_\_\_]</a>\"
```

Usage of Regular Expressions in Computer Security

- In computer security one very often needs to check for occurrences of certain patterns.
- For instance in order to locate a certain virus, which might consist of 3 pieces of code \(s_1, s_2, s_3\), separated by some normal code, one could search for the regular expression

\[s_1[a-z]^*s_2[a-z]^*s_3\]

(How do you obtain that \(s_1, s_2\) and \(s_3\) might occur in different order?)
- Of course in general you need to check for much more sophisticated patterns.
In order to check that a password is safe enough, which might mean it consists of digits and lower case characters, and at least one digit and one lower case character, would mean that you whether it matches

$$((a - z) | [0 - 9])*([a - z] | [0 - 9])*([a - z] | [0 - 9])^*) |$$

$$((a - z) | [0 - 9])*([0 - 9])^*([a - z] | [0 - 9])^*([a - z])^*([0 - 9])^*$$

Of course you would usually use a much more sophisticated regular expression.

Detecting in request certain malicious patterns in requests from the outside can often be expressed as a regular expression and you search for matches in this income stream which match that expression.

SQL Injection

- Regular expression can be used to detect attempts of SQL injection.
- Example of SQL injection (from Wikipedia on SQL Injection):
  Assume the following statement in a code

  ```
  statement =
  "SELECT * FROM users WHERE name =' " + userName + " ';"
  ```

- This statement is supposed to be sent to the SQL server. Then one checks the resulting entries for whether the supplied password matches modulo encryption one of the password entries for that user name.
- Assume an attacker tries to login with user name
  `' or '1'='1`

  Then the statement sent to the SQL server will be

  ```
  SELECT * FROM users WHERE name ='' or '1'='1';
  ```

  which matches all users.

- This might allow you to check whether your password matches any user, which makes it more likely to get a match and allow you to login.
- In order to avoid such kind of attack you can check whether the username matches any malicious pattern. Such patterns can be expressed by regular expressions.
The above were just some (here very simple) examples how regular expressions can be used to detect in computer security certain patterns corresponding to attacks or weaknesses of a system.

**Closure of Regular Languages**

The main lemma for showing that regular expressions define regular languages is as follows:

**Lemma (II.2.2.1.)**

Let $G, G'$ be both left-linear grammars or both right-linear grammars. Then we can define a left-linear or right-linear grammars $G_i$ s.t.

1. $L(G_1) = L(G) | L(G')$,
2. $L(G_2) = L(G).L(G')$,
3. $L(G_3) = L(G)^*$.

These grammars can be computed from $G$ and $G'$.

**Proof**

A proof can be found in the additional material for this subsection.

**Regular Expressions define Regular Languages**

**Lemma (II.2.2.2.)**

Let $E$ be a regular Expression. Then there exist both left-linear and right-linear grammars $G, G'$ s.t.

$$L(E) = L(G) = L(G')$$

$G$ and $G'$ can be computed from $L$.

**Proof:** By Lemma II.2.2.1, and the fact that the finite languages $\emptyset, \{\epsilon\}$ and $\{a\}$ are regular.

Full details can be found in Additional Material.