III.2 (a) Definition of the URM

III.2 (b) Higher level programming concepts for URMs

III.2 (c) URM computable functions
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III.2 (c) URM computable functions
No Additional Material

For this subsection no additional material has been added yet.
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III.2 (a) Definition of the URM

III.2 (b) Higher level programming concepts for URMs

III.2 (c) URM computable functions
III.2 (c) URM-Computable Functions

- We introduce some constructions for introducing URM-computable functions.
- We will later introduce the set of partial recursive functions as the least set of functions closed under these constructions.
  - Then by the fact that the URM-computable functions are closed under these operations it follows that all partial recursive functions are URM-computable.
- We introduce first names for all functions constructed this way.
Definition 2.1

(a) Define the **zero function** \( \text{zero} : \mathbb{N} \to \mathbb{N} \), \( \text{zero}(x) = 0 \).

(b) Define the **successor function** \( \text{succ} : \mathbb{N} \to \mathbb{N} \), \( \text{succ}(x) = x + 1 \).

(c) Define for \( 0 \leq i < n \) the **projection function** \( \text{proj}^n_i : \mathbb{N}^n \to \mathbb{N} \),

\[
\text{proj}^n_i(x_0, \ldots, x_{n-1}) = x_i.
\]

**Remark**

- Note that all total functions are as well partial, so we have for instance as well \( \text{zero} : \mathbb{N} \xrightarrow{\sim} \mathbb{N} \).

- \( \text{proj}^1_0 : \mathbb{N} \to \mathbb{N} \) is the identity function: \( \text{proj}^1_0(x) = x \).
III.2 (c) URM computable functions

Notations for Partial Functions

Definition (Cont)

(d) Assume

\[ g : (B_0 \times \cdots \times B_{k-1}) \sim \rightarrow C, \]
\[ h_i : A_0 \times \cdots \times A_{n-1} \sim \rightarrow B_i, \quad i = 0, \ldots, k - 1 \]

Define

\[ f := g \circ (h_0, \ldots, h_{k-1}) : A_0 \times \cdots \times A_{n-1} \sim \rightarrow C : \]

\[ f(\vec{a}) \sim g(h_0(\vec{a}), \ldots, h_{k-1}(\vec{a})) \]
Notations for Partial Functions

Definition (Cont)

- In case of \( k = 1 \) we write \( g \circ h \) instead of \( g \circ (h) \).
- Furthermore as usual

\[
g_1 \circ g_2 \circ \cdots \circ g_n \equiv g_1 \circ (g_2 \circ (\cdots \circ (g_{n-1} \circ g_n)))
\]
(e) Assume

\[ g : \mathbb{N}^k \rightarrow \mathbb{N}, \]
\[ h : \mathbb{N}^{k+2} \rightarrow \mathbb{N}. \]

Then we can define a function \( f : \mathbb{N}^{k+1} \rightarrow \mathbb{N} \) defined by \textbf{primitive recursion} from \( g \) and \( h \) as follows:

\[
\begin{align*}
f(\vec{n}, 0) & : \equiv g(\vec{n}) \\
f(\vec{n}, m + 1) & : \equiv h(\vec{n}, m, f(\vec{n}, m))
\end{align*}
\]

\textbf{We write} \texttt{primrec}(g, h) \textbf{ for the function } f \textbf{ just defined.}

\textbf{So} \texttt{primrec}(g, h) : \mathbb{N}^{k+1} \rightarrow \mathbb{N}.
In the special case $k = 0$, it doesn’t make sense to use $g()$. Instead replace in this case $g$ by some natural number. So the case $k = 0$ reads as follows:

Assume $a \in \mathbb{N}$, $h : \mathbb{N}^2 \sim \rightarrow \mathbb{N}$.

Define

$$f : \mathbb{N} \sim \rightarrow \mathbb{N}$$

by primitive recursion from $a$ and $h$ as follows:

$$f(0) : \simeq a$$
$$f(m + 1) : \simeq h(m, f(m))$$

We write $\text{primrec}(a, h)$ for $f$, so $\text{primrec}(a, h) : \mathbb{N} \sim \rightarrow \mathbb{N}$. 
primrec in Haskell

- In Haskell we can define `primrec` as a higher-order function as follows:
  
  data Nat = Z | S Nat
  deriving Show

  - `primrec0` is the operator for primitive recursion
  - defining a 1-ary function `primrec0 f a :: Nat → Nat`
  - from `f: Nat → Nat → Nat` and `a: Nat`

  `primrec0 :: Nat → (Nat → Nat → Nat) → Nat → Nat`
  `primrec0 a g Z = a`
  `primrec0 a g (S n) = g n (primrec0 a g n)`
- - primrec1 is the operator for primitive recursion
- - defining a 2-ary function \( \text{primrec1} \ f \ g \) :: Nat \( \rightarrow \) Nat \( \rightarrow \) Nat
- - from \( f : \text{Nat} \rightarrow \text{Nat} \rightarrow \text{Nat} \rightarrow \text{Nat} \) and \( g : \text{Nat} \rightarrow \text{Nat} \)

\[
\text{primrec1} :: (\text{Nat} \rightarrow \text{Nat})
\rightarrow (\text{Nat} \rightarrow \text{Nat} \rightarrow \text{Nat} \rightarrow \text{Nat})
\rightarrow \text{Nat} \rightarrow \text{Nat} \rightarrow \text{Nat}
\]

\[
\text{primrec1} \ g \ h \ n \ Z = g \ n
\]

\[
\text{primrec1} \ g \ h \ n \ (S \ m) = h \ n \ m \ (\text{primrec1} \ g \ h \ n \ m)
\]
Addition can be defined using primitive recursion:

Let $\text{add} : \mathbb{N}^2 \to \mathbb{N}$, $\text{add}(x, y) := x + y$. We have

\[
\begin{align*}
\text{add}(x, 0) &= x + 0 = x \\
\text{add}(x, y + 1) &= x + (y + 1) = (x + y) + 1 = \text{add}(x, y) + 1
\end{align*}
\]

Therefore

\[
\begin{align*}
\text{add}(x, 0) &= g(x) \\
\text{add}(x, y + 1) &= h(x, y, \text{add}(x, y))
\end{align*}
\]

where

\[
\begin{align*}
g : \mathbb{N} \to \mathbb{N} , & \quad g(x) := x \, , \\
h : \mathbb{N}^3 \to \mathbb{N} , & \quad h(x, y, z) := z + 1 .
\end{align*}
\]

So $\text{add} = \text{primrec}(g, h)$. 
Addition (add)

\[ g : \mathbb{N} \to \mathbb{N} , \quad g(x) := x , \]
\[ h : \mathbb{N}^3 \to \mathbb{N} , \quad h(x, y, z) := z + 1 , \]
\[ \text{add} := \text{primrec}(g, h) \]

- We have
  - \( \text{add}(x, 0) = g(x) = x = x + 0. \)
  - \( \text{add}(x, 1) = h(x, 0, \text{add}(x, 0)) = \text{add}(x, 0) + 1 = x + 1. \)
  - \( \text{add}(x, 2) = h(x, 1, \text{add}(x, 1)) = \text{add}(x, 1) + 1 = (x + 1) + 1. \)
  - etc.
In Haskell we can define add from primrec as follows

```
add :: Nat → Nat → Nat
add = primrec1 (λ n → n) (λ n m k → S k)
```
Examples for Primitive Recursion

- Multiplication can be defined using primitive recursion:
  Let \( \text{mult} : \mathbb{N}^2 \rightarrow \mathbb{N}, \text{mult}(x, y) := x \cdot y. \) We have

  \[
  \begin{align*}
  \text{mult}(x, 0) &= x \cdot 0 = 0 \\
  \text{mult}(x, y + 1) &= x \cdot (y + 1) = x \cdot y + x = \text{mult}(x, y) + x
  \end{align*}
  \]

  Therefore

  \[
  \begin{align*}
  \text{mult}(x, 0) &= g(x) \\
  \text{mult}(x, y + 1) &= h(x, y, \text{mult}(x, y))
  \end{align*}
  \]

  where

  \[
  \begin{align*}
  g : \mathbb{N} &\rightarrow \mathbb{N} , & g(x) := 0 , \\
  h : \mathbb{N}^3 &\rightarrow \mathbb{N} , & h(x, y, z) := z + x .
  \end{align*}
  \]

  So \( \text{mult} = \text{primrec}(g, h). \)
Multiplication (mult)

\[ g : \mathbb{N} \rightarrow \mathbb{N} , \quad g(x) := 0 , \]
\[ h : \mathbb{N}^3 \rightarrow \mathbb{N} , \quad h(x, y, z) := z + x , \]
\[ \text{mult} := \text{primrec}(g, h) \]

- We have
  - \[ \text{mult}(x, 0) = g(x) = 0 = x \cdot 0. \]
  - \[ \text{mult}(x, 1) = h(x, 0, \text{mult}(x, 0)) = \text{mult}(x, 0) + x = 0 + x = x. \]
  - \[ \text{mult}(x, 2) = h(x, 1, \text{mult}(x, 1)) = \text{mult}(x, 1) + x = (x \cdot 1) + x. \]
  - etc.
Examples for Primitive Recursion

Let \( \text{pred} : \mathbb{N} \rightarrow \mathbb{N} \), \( \text{pred}(n) := n - 1 = \begin{cases} n - 1 & \text{if } n > 0, \\ 0 & \text{otherwise}. \end{cases} \)

\( \text{pred} \) can be defined using primitive recursion:

\[
\begin{align*}
\text{pred}(0) &= 0 \\
\text{pred}(x + 1) &= x
\end{align*}
\]

Therefore

\[
\begin{align*}
\text{pred}(0) &= 0 \\
\text{pred}(x + 1) &= h(x, \text{pred}(x))
\end{align*}
\]

where

\[
h : \mathbb{N}^2 \rightarrow \mathbb{N}, \quad h(x, y) := x
\]

So \( \text{pred} = \text{primrec}(0, h) \).
Examples for Primitive Recursion

- $x - y$ can be defined using primitive recursion:
  Let $f(x, y) := x - y$. We have

  \[
  f(x, 0) = x - 0 = x \\
  f(x, y + 1) = x - (y + 1) = (x - y) - 1 \\
  = \text{pred}(x - y) = \text{pred}(f(x, y))
  \]

  Therefore

  \[
  f(x, 0) = g(x) \\
  f(x, y + 1) = h(x, y, f(x, y))
  \]

  where

  \[
  g: \mathbb{N} \to \mathbb{N} , \quad g(x) := x , \\
  h: \mathbb{N}^3 \to \mathbb{N} , \quad h(x, y, z) := \text{pred}(z) .
  \]

  So $f = \text{primrec}(g, h)$. 
Remark

- If \( f = \text{primrec}(g, h) \), then

\[
f(\vec{n}, m) \uparrow \rightarrow \forall k \geq m. f(\vec{n}, k) \uparrow
\]

- **Proof:**
  - We have

\[
f(\vec{n}, m + 1) \simeq h(\vec{n}, m, f(\vec{n}, m))
\]
  - All functions are strict.
  - So if \( f(\vec{n}, m) \uparrow \), then

\[
f(\vec{n}, m + 1) \simeq h(\vec{n}, m, f(\vec{n}, m)) \uparrow
\]

therefore

\[
f(\vec{n}, m + 1) \uparrow
\]
Proof of Remark

- Therefore we have

\[ f(\vec{n}, m) \uparrow \rightarrow f(\vec{n}, m + 1) \uparrow. \]

- By induction it follows that \( f(\vec{n}, m) \uparrow \) implies

\[ \forall k \geq m. f(\vec{n}, k) \uparrow. \]
Example

Let

\[ h : \mathbb{N}^2 \rightarrow \mathbb{N}, \quad h(n, m) \simeq \begin{cases} \frac{m - 1}{1} & \text{if } m > 0, \\ \bot & \text{otherwise.} \end{cases} \]

Let

\[ f : \mathbb{N} \rightarrow \mathbb{N} \], \quad f := \text{primrec}(1, h), \]
i.e. \[ f(0) \simeq 1 \], \quad f(n + 1) \simeq h(n, f(n)). \]

Then

\[ f(0) \simeq 1 \]
\[ f(1) \simeq h(0, f(0)) \simeq h(0, 1) \simeq 0 \]
\[ f(2) \simeq h(1, f(1)) \simeq h(1, 0) \uparrow \]
\[ \forall m \geq 2. f(m) \uparrow \]
The functions, which can be defined from zero, succ, proj\textsubscript{k} by using composition (\(\circ\)) and primitive recursion (primrec) are called the 
\textbf{primitive recursive functions}.

The primitive-recursive functions will be studied more in detail in Sect. 5.

There we will see that they are powerful, but \textbf{not Turing-complete}.
In many expressions we will have arguments, to which we don’t refer explicitly.

**Example:** Variables $x_0, \ldots, x_{n-1}$ in

$$f(x_0, \ldots, x_{n-1}, y) = \begin{cases} g(x_0, \ldots, x_{n-1}), & \text{if } y = 0, \\ h(x_0, \ldots, x_{n-1}), & \text{if } y > 0. \end{cases}$$

We abbreviate $x_0, \ldots, x_{n-1}$, by $\vec{x}$.

Then the above can be written shorter as

$$f(\vec{x}, y) = \begin{cases} g(\vec{x}), & \text{if } y = 0, \\ h(\vec{x}), & \text{if } y > 0. \end{cases}$$

In general, $\vec{x}$ stands for $x_0, \ldots, x_{n-1}$, where the number of arguments $n$ is clear from the context.
Examples

- If
  \[ f : \mathbb{N}^{n+1} \rightarrow \mathbb{N} \]
  then in \( f(\vec{x}, y) \),
  \( \vec{x} \) needs to stand for \( n \) arguments.
  Therefore
  \[ \vec{x} = x_0, \ldots, x_{n-1} \]

- If
  \[ f : \mathbb{N}^{n+2} \rightarrow \mathbb{N} \]
  then in \( f(\vec{x}, y) \),
  \( \vec{x} \) needs to stand for \( n + 1 \) arguments,
  so
  \[ \vec{x} = x_0, \ldots, x_n \]
Examples

- If $P$ is an $n + 4$-ary relation, then in $P(\vec{x}, y, z)$, $
\vec{x}$ stands for $\ x_0, \ldots, x_{n+1}$

- Similarly, we write $\vec{y}$ for $\ y_0, \ldots, y_{n-1}$

  where $n$ is clear from the context.

- Similarly for $\vec{z}, \vec{n}, \vec{m}, \ldots$
Notation

\( \forall \bar{x} \in \mathbb{N}. \varphi(\bar{x}) \)

stands for

\( \forall x_0, \ldots, x_{n-1} \in \mathbb{N}. \varphi(x_0, \ldots, x_{n-1}) \)

where the number of variables \( n \) is implicit (and usually unimportant).

\( \exists \bar{x} \in \mathbb{N}. \varphi(\bar{x}) \)

is to be understood similarly.
Notation

\[
\{ \vec{x} \in \mathbb{N}^n | \varphi(\vec{x}) \}\]

is to be understood as
\[
\{(x_0, \ldots, x_{n-1}) \in \mathbb{N}^n | \varphi(x_0, \ldots, x_{n-1}) \}\]

\[
\{ (\vec{x}, y, z) \in \mathbb{N}^{n+2} | \varphi(\vec{x}, y, z) \}\]

is to be understood as
\[
\{(x_0, \ldots, x_{n-1}, y, z) \in \mathbb{N}^{n+2} | \varphi(x_0, \ldots, x_{n-1}, y, z) \}\]

Similar notations are to be understood analogously.
Let \( g : \mathbb{N}^{n+1} \rightarrow \mathbb{N} \).

We define \( \mu y . (g(\vec{x}, y) \simeq 0) \)

(Here \( \vec{x} \) stands for arguments \( x_1, \ldots, x_n \)).

\[
\mu y . (g(\vec{x}, y) \simeq 0) : \simeq \begin{cases} 
    \text{the least } y \in \mathbb{N} \text{ s.t.} \\
    g(\vec{x}, y) \simeq 0 \\
    \text{and for } 0 \leq y' < y \\
    \text{there exists a } z' \neq 0 \\
    \text{s.t. } g(\vec{x}, y') \simeq z' & \text{if such } y \text{ exists,} \\
    \bot & \text{otherwise}
\end{cases}
\]
Definition (Cont)

- Now define $h : \mathbb{N}^n \sim \mathbb{N}$,

\[ h(\vec{x}) \simeq \mu y. (g(\vec{x}, y) \simeq 0) \]

- We write $\mu(g)$ for this function $h$. 

$\mu(g)$
Examples

- Assume

\[ g(x, 0) \simeq 1 \]
\[ g(x, 1) \uparrow \]
\[ g(x, 2) \simeq 0 \]

Then

\[ \mu y. (g(x, y) \simeq 0) \uparrow \]

- Assume instead

\[ g(x, 0) \simeq 1 \]
\[ g(x, 1) \simeq 5 \]
\[ g(x, 2) \simeq 0 \]

Then

\[ \mu y. (g(x, y) \simeq 0) \simeq 2 \]
Computation of $\mu(g)$

$\mu(g)(\vec{x}) : \simeq \mu y.(g(\vec{x}, y) \simeq 0)$.

$\Rightarrow$ If $g$ is intuitively computable, we see that $h := \mu(g)$ is intuitively computable as follows:

$\Rightarrow$ In order to compute $h(\vec{x})$ we first compute $g(\vec{x}, 0)$.

$\Rightarrow$ If this computation never terminates $g(\vec{x}, 0) \uparrow$ and $\mu y.(g(\vec{x}, y) \simeq 0) \uparrow$ as well.

$\Rightarrow$ If it terminates, and we have $g(\vec{x}, 0) \simeq 0$, we obtain $\mu y.(g(\vec{x}, y) \simeq 0) \simeq 0$.

$\Rightarrow$ Otherwise, repeat the above with testing of $g(\vec{x}, 1) \simeq 0$.

$\Rightarrow$ If successful $\mu y.(g(\vec{x}, y) \simeq 0) \simeq 1$.

$\Rightarrow$ If unsuccessful repeat it with 2, 3, etc.
Computation of $\mu(g)$

- Note that $\mu(g)(\vec{x}) \uparrow$ in case there is a $y$ s.t.
  - $g(\vec{x}, y) \uparrow$
  - and for $y' < y$ we have $g(\vec{x}, y') \downarrow$ but $g(\vec{x}, y') \simeq z$ for some $z > 0$.
- This coincides with computation by the above mentioned intuitive computation:
  - In this case, the program will compute $g(\vec{x}, 0), g(\vec{x}, 1), \ldots, g(\vec{x}, y - 1)$ and get as result that these values are $\neq 0$.
  - Then it will try to compute $g(\vec{x}, y)$, and this computation never terminates.
  - So the value of this program is undefined, as is $\mu y.(g(\vec{x}, y) \simeq 0)$. 
Computation of $\mu(g)$

- If we defined $\mu(g)(\vec{x})$ to be the least $y$ s.t.

$$g(\vec{x}, y) \simeq 0$$

independently of whether $g(\vec{x}, y') \downarrow$ for all $y' < y$, then we would obtain a **non computable function**.
Examples for $\mu$

- Let $f : \mathbb{N}^2 \rightarrow \mathbb{N}$, $f(x, y) := x \div y$. Then
  \[ \mu y. (f(x, y) \simeq 0) \simeq x \]
  so $\mu(f)(x) \simeq x$.

- Let $f : \mathbb{N} \rightarrow \mathbb{N}$,
  $f(0) \uparrow$,
  $f(n) := 0$ for $n > 0$.
  Then
  \[ \mu y. (f(y) \simeq 0) \uparrow \].
Examples for $\mu$

Let $f : \mathbb{N} \rightarrow\rightarrow \mathbb{N}$,

$$f(n) := \begin{cases} 1 & \text{if there exist primes } p, q < 2n + 4 \\ s.t. \ 2n + 4 = p + q, \\ 0 & \text{otherwise} \end{cases}$$

$\mu y.(f(y) \simeq 0)$ is the first $n$ s.t. there don’t exist primes $p, q$ s.t. $2n + 4 = p + q$.

**Goldbach’s conjecture** says that every even number $\geq 4$ is the sum of two primes.

This is equivalent to $\mu y.('f(y) \simeq 0)\uparrow$.

It is one of the most important open problems in mathematics to show (or refute) Goldbach’s conjecture.

If we could decide whether a partial computing function is defined (which we can’t), we could decide Goldbach’s conjecture.
Partial Recursive Functions

- The functions, which can define in the same way as the primitive-recursive functions
  - i.e. being defined from zero, succ, proj\textsubscript{k} by using composition (\circ) and primitive recursion (primrec)

but by additionally closing them under \( \mu \), are called the **partial recursive functions**.

- The partial recursive functions will be studied more in detail in Sect. 6.
  - There we will see that the partial recursive functions **form a Turing complete model of computation**.
Next Step

- We are going to show that the URM computable functions are closed under the operations introduced above.
- In order to show this we need to be able to modify URM programs, so that they
  - have some other specified input and output registers,
  - and conserve the content of certain other registers.
- The following lemma shows that such a modification is possible.
Lemma and Definition 2.2

Assume \( f : \mathbb{N}^k \xrightarrow{\sim} \mathbb{N} \) is URM-computable.
Assume \( x_0, \ldots, x_{k-1}, y, z_0, \ldots, z_l \) are different variables.
Then one can define a URM program, which, computes \( f(x_0, \ldots, x_{k-1}) \) and stores the result in \( y \) in the following sense:

- If \( f(x_0, \ldots, x_{k-1}) \downarrow \), the program terminates at the first instruction following this program, and stores the result in \( y \).
- If \( f(x_0, \ldots, x_{k-1}) \uparrow \), the program never terminates.

The program can be defined so that it doesn’t change \( x_0, \ldots, x_{k-1}, z_0, \ldots, z_l \).
For \( \mathbb{U} \) we say it is a URM program which computes \( y \approx f(x_0, \ldots, x_{k-1}) \) and preserves \( z_0, \ldots, z_l \).
Intuition behind Lem. 2.2

Lemma 2.2 means that if $f$ is URM-computable then we can define a URM-program in such a way that

- it takes the arguments from registers we have chosen,
- and stores the result in a register we have chosen,
- and does this in such a way that the content of the input registers and of some other registers we have chosen are not modified.
- This is possible as long as the input registers and the output register are all different.
Idea of the proof

- First copy the arguments in some other registers, so that the arguments are preserved.
- Then compute the function on those auxiliary registers and make sure that the computation doesn’t affect the registers to be preserved.
- Then move the result into the register chosen as output register, and set variables $x_0, \ldots, x_{k-1}, z_0, \ldots, z_l$ back to their original (stored) values.

Omit Proof.
Proof

Let $U$ be a URM program s.t. $U^{(k)} = f$.
Let $u_0, \ldots, u_{k-1}$ be registers different from the above.
By renumbering of registers and of jump addresses, we obtain a program $U'$, which computes the result of $f(u_0, \ldots, u_{k-1})$ in $u_0$.
leaves the registers mentioned in the lemma unchanged,
and which, if it terminates, terminates in the first instruction following $U'$.
The following is a program as intended:

\[
\begin{align*}
u_0 &:= x_0; \\
\cdots \\
u_{k-1} &:= x_{k-1}; \\
U' &
y := u_0;
\end{align*}
\]
Lemma 2.3

1. zero, succ and $\text{proj}^n_i$ are URM-computable.
2. If $f : \mathbb{N}^n \xrightarrow{\sim} \mathbb{N}$, $g_i : \mathbb{N}^k \xrightarrow{\sim} \mathbb{N}$ are URM-computable, so is $f \circ (g_0, \ldots, g_{n-1})$.
3. If $g : \mathbb{N}^n \xrightarrow{\sim} \mathbb{N}$, and $h : \mathbb{N}^{n+2} \xrightarrow{\sim} \mathbb{N}$ are URM-computable, so is the function $f := \text{primrec}(g, h)$ defined by primitive recursion from $g$ and $h$.
4. If $g : \mathbb{N}^{n+1} \xrightarrow{\sim} \mathbb{N}$ is URM-computable, so is $\mu(g)$. 
Remark

- The Lemma is very powerful:
  - It shows that many functions are URM-computable.
  - This shows that for instance the exponential function is URM computable.
    - This follows since addition, multiplication and exponentiation can be defined by primitive recursion from the basic functions.
    - Writing a URM program directly which computes the exponential function would be very difficult.

**Omit Proof.**
Proof of Lemma 2.3 (a)

Let $x_i$ denote register $R_i$.

**Proof of (a)**

- zero is computed by the following program:
  $$x_0 := 0.$$

- succ is computed by the following program:
  $$x_0 := x_0 + 1.$$

- $\text{proj}_n^k$ is computed by the following program:
  $$x_0 := x_k.$$
  Especially, if $k = 0$ then $\text{proj}_n^k$ is the empty program (i.e. the program with no instructions this is since we defined $x_0 := x_0$ to be the empty program.)
Proof of Lemma 2.3 (b)

Assume \( f : \mathbb{N}^n \rightarrow \mathbb{N} \), \( g_i : \mathbb{N}^k \rightarrow \mathbb{N} \) are URM-computable.

Show \( f \circ (g_0, \ldots, g_{n-1}) \) is computable.

A plan for the program is as follows:

- **Input** is stored in registers \( x_0, \ldots, x_{k-1} \).
  Let \( \vec{x} := x_0, \ldots, x_{k-1} \).

- First we compute \( g_i(\vec{x}) \) for \( i = 0, \ldots, n-1 \), store result in registers \( y_i \).
  - By Lemma 2.2 we can do this in such a way that \( x_0, \ldots, x_{k-1} \) and the previously computed values \( g_i(\vec{x}) \), which are stored in \( y_j \) for \( j < i \) are not destroyed.

- Then compute \( f(y_0, \ldots, y_{n-1}) \), and store result in \( x_0 \).

- Then \( x_0 \) contains \( f(g_0(\vec{x}), \ldots, g_{n-1}(\vec{x})) \).
Proof of Lemma 2.3 (b)

Let therefore $U_i$ be a URM program ($i = 0, \ldots, n - 1$), which computes $y_i \simeq g_i(\vec{x})$ and preserves $y_j$ for $j \neq i$.

Let $V$ be a URM program, which computes $x_0 \simeq f(y_0, \ldots, y_{n-1})$. 
Proof of Lemma 2.3 (b)

Let $U'$ be defined as follows:

$U_0$

$\ldots$

$U_{n-1}$

$V$

We show $U'_{(k)}(\bar{x}) \simeq (f \circ (g_0(\bar{x}), \ldots, g_{n-1}(\bar{x}))).$

Omit rest of proof.
Proof of Lemma 2.3 (b)

$U'$ is the program

$U_0$

$\ldots$

$U_{n-1}$

$V$

- **Case 1:** For one $i \ g_i(\bar{x}) \uparrow$.
  
The program will loop in program $U_i$ for the first such $i$.
  
  $U'(k)(\bar{x}) \uparrow$, $f \circ (g_0, \ldots, g_{n-1})(\bar{x}) \uparrow$.

- **Case 2:** For all $i \ g_i(\bar{x}) \downarrow$.
  
The program executes $U_i$, sets $y_i \simeq g_i(x_0, \ldots, x_{k-1})$ and reaches beginning of $V$. 
Proof of Lemma 2.3 (b)

$U'$ is the program

$U_0$

$\ldots$

$U_{n-1}$

$V$

- **Case 2.1:** $f(g_0(\vec{x}), \ldots, g_{n-1}(\vec{x})) \uparrow$.  
  $V$ will loop, $U'^{(k)}(\vec{x}) \uparrow, f \circ (g_0, \ldots, g_{n-1})(\vec{x}) \uparrow$.

- **Case 2.2:** Otherwise.
  The program reaches the end of program $V$ and result in
  $x_0 \simeq f(g_0(\vec{x}), \ldots, g_{n-1}(\vec{x}))$.
  So $U'^{(k)}(\vec{x}) \simeq (f \circ (g_0, \ldots, g_{n-1}))(\vec{x})$. 
Proof of Lemma 2.3 (b)

In all cases

\[ U'(k)(\bar{x}) \simeq (f \circ (g_0, \ldots, g_{n-1}))(\bar{x}). \]
Proof of Lemma 2.3 (c)

Assume

\[ g : \mathbb{N}^n \rightarrow\!\!\!\!\!\!\sim \mathbb{N} , \quad h : \mathbb{N}^{n+2} \rightarrow\!\!\!\!\!\!\sim \mathbb{N} \]

are URM-computable.

Let

\[ f := \text{primrec}(g, h) . \]

Show \( f \) is URM-computable.

Defining equations for \( f \) are as follows

(let \( \vec{n} := n_0, \ldots, n_{n-1} \)):

- \( f(\vec{n}, 0) \simeq g(\vec{n}) \),
- \( f(\vec{n}, k + 1) \simeq h(\vec{n}, k, f(\vec{n}, k)) \).
Proof of Lemma 2.3 (c)

Computation of $f(\vec{n}, l)$ for $l > 0$ is as follows:

- Compute $f(\vec{n}, 0)$ as $g(\vec{n})$.
- Compute $f(\vec{n}, 1)$ as $h(\vec{n}, 0, f(\vec{n}, 0))$, using the previous result.
- Compute $f(\vec{n}, 2)$ as $h(\vec{n}, 1, f(\vec{n}, 1))$, using the previous result.
- ... 
- Compute $f(\vec{n}, l)$ as $h(\vec{n}, l - 1, f(\vec{n}, l - 1))$, using the previous result.
Proof of Lemma 2.3 (c)

Plan for the program:

- Let $\vec{x} := x_0, \ldots, x_{n-1}$.
  Let $y, z, u$ be new registers.
- Compute $f(\vec{x}, y)$ for $y = 0, 1, 2, \ldots, x_n$, and store result in $z$.
  - Initially we have $y = 0$ (holds for all registers except of $x_0, \ldots, x_n$ initially).
    We compute $z \simeq g(\vec{x}) \ (\simeq f(\vec{x}, 0))$.
    Then $y = 0, z \simeq f(\vec{x}, 0)$. 
Proof of Lemma 2.3 (c)

In step from $y$ to $y + 1$:

- Assume that we have $z \simeq f(\bar{x}, y)$.
- We want that after increasing $y$ by 1 the loop invariant $z \simeq f(\bar{x}, y)$ still holds.

Obtained as follows:

- Compute $u \simeq h(\bar{x}, y, z) \simeq h(\bar{x}, y, f(\bar{x}, y)) \simeq f(\bar{x}, y + 1)$.
- Execute $z := u \simeq f(\bar{x}, y + 1)$.
- Execute $y := y + 1$.
- At the end, $z \simeq f(\bar{x}, y)$ for the new value of $y$.

- Repeat this until $y = x_n$.

Once $y$ has reached $x_n$, $z$ contains $f(\bar{x}, y) \simeq f(\bar{x}, x_n)$.

- Execute $x_0 := z$. 
Proof of Lemma 2.3 (c)

Let

- $U$ be a URM program, which computes $z \simeq g(\vec{x})$ and preserves $y$ (by definition 2.2, it doesn’t modify the arguments $\vec{x}$ of $g$);
- $V$ be a program, which computes $u \simeq h(\vec{x}, y, z)$. (by definition 2.2, it doesn’t change $\vec{x}, y, z$.)
Proof of Lemma 2.3 (c)

Let $U'$ be as follows:

$U$

while ($x_n \neq y$) do {
  $V$

  $z := u$;
  $y := y + 1$;
}

$x_0 := z$;

$$z \leftarrow g(\vec{x}) (\sim f(\vec{x}, 0))$$

$$u \leftarrow h(\vec{x}, y, z)$$

will be $h(\vec{x}, y, f(\vec{x}, y)) \leftarrow f(\vec{x}, y + 1)$
Proof of Lemma 2.3 (c)

Correctness of this program:

- When $U$ has terminated, we have $y = 0$ and $z \simeq g(\vec{x}) \simeq f(\vec{x}, y)$.
- After each iteration of the while loop, we have $y := y' + 1$ and $z \simeq h(\vec{x}, y', z')$.  
  ($y'$, $z'$ are the previous values of $y$, $z$, respectively.)
- Therefore we have $z \simeq f(\vec{x}, y)$.
- The loop terminates, when $y$ has reached $x_n$. Then $z$ contains $f(\vec{x}, y)$. This is stored in $x_0$. 
Proof of Lemma 2.3 (c)

- If $U$ loops for ever, or in one of the iterations $V$ loops for ever, then:
  - $U'$ loops, $U'^{(n+1)}(\vec{x}, x_n) \uparrow$.
  - $f(\vec{x}, k) \uparrow$ for some $k < x_n$.
  - Subsequently $f(\vec{x}, l) \uparrow$ for all $l > k$.
  - Especially, $f(\vec{x}, x_n) \uparrow$.
  - Therefore $f(\vec{x}, x_n) \simeq U'^{(n+1)}(\vec{x}, x_n)$. 
Proof of Lemma 2.3 (d)

Assume

\[ g : \mathbb{N}^{n+1} \sim \rightarrow \mathbb{N} \]

is URM-computable.

Show

\[ \mu(g) \]

is URM-computable as well.

Note \( \mu(g)(x_0, \ldots, x_{k-1}) \) is the minimal \( z \) s.t.

\[ g(x_0, \ldots, x_{k-1}, z) \simeq 0 \, . \]

Let \( \vec{x} := x_0, \ldots, x_{k-1} \) and let \( y, z \) be registers different from \( \vec{x} \).
Proof of Lemma 2.3 (d)

Plan for the program:

- Compute $g(\vec{x}, 0), g(\vec{x}, 1), \ldots$ until we find a $k$ s.t. $g(\vec{x}, k) \simeq 0$. Then return $k$.
- This is carried out by executing

$$z \simeq g(\vec{x}, y)$$

and successively increasing $y$ by 1 until we have $z = 0$. 

Proof of Lemma 2.3 (d)

Let $U$ compute

$$z \simeq g(x_0, \ldots, x_{k-1}, y),$$

(and preserve the arguments $x_0, \ldots, x_{k-1}, y$.)

Let $V$ be as follows:

```
repeat{
    U
    y := y + 1;
} until (z = 0);

y := y - 1;
x_0 := y;
```

Omit rest of proof.
Proof of Lemma 2.3 (d)

\[ V \text{ is } \text{repeat}\{U; y := y + 1;\} \text{ until } (z = 0); \]
\[ y := y \div 1; x_0 := y; \]

Initially \( y = 0 \).

After each iteration of the repeat loop, we have

\[ y := y' + 1, \quad z \simeq g(x_0, \ldots, x_{k-1}, y') \]

\((y' \text{ is the value of } y \text{ before this iteration})\).

If the loop terminates, we have

\[ z \simeq 0 \quad y = y' + 1 \]

where \( y' \text{ is the first value, such that } g(x_0, \ldots, x_{k-1}, y') \simeq 0 \).
Proof of Lemma 2.3 (d)

- Finally $y$ is decreased by one.
- Then $y$ is the least $y$ s.t.
  \[ g(x_0, \ldots, x_{k-1}, y) \simeq 0. \]
- $x_0$ is then set to that value.